Introduction

Thank you for using ARCH as you prepare for Course M. We hope you find our material helpful.

As you work through these problems, I don't think you'll find any of the math to be difficult (or even new). The tricky parts of these questions are:

- understanding what the question is asking. Sketch a time-line or decision tree to help you get a good understanding of the different potential paths, cash flows, etc.
- reading matrices correctly. Illustrative matrices similar to these could very well make their way onto the exam.
- accurately performing the calculations, particularly matrix algebra. Be careful, but be quick!

CHAPTER 1 – MULTI-STATE TRANSITION MODELS FOR ACTUARIAL APPLICATIONS

1. By Definition 1.15, we're looking for

$$\mathbf{Q_{30}} = \left\| \begin{matrix} p_{60+30} & q_{60+30} \\ 0 & 1 \end{matrix} \right\|$$

For DeMoivre, we have

$$_tp_x = \frac{\omega - x - t}{\omega - x} = 1 - {_tq_x}$$

$$p_{60+30} = \frac{100 - 90 - 1}{100 - 90} = 0.9$$

$$q_{60+30} = 1 - p_{60+30} = 0.1$$

So we have

$$\mathbf{Q_{30}} = \left\| \begin{matrix} 0.9 & 0.1 \\ 0 & 1 \end{matrix} \right\|$$

2. From Example 1.10,

$$Q_{10}^{(1,2)} = p_{x+10} \cdot q_{x+10}$$

So we need p_{70} and q_{85}. For DeMoivre, we have

$$p_{70} = \frac{100 - 70 - 1}{100 - 70} = \frac{29}{30}$$

Likewise,

$$q_{85} = \frac{t}{\omega - x} = \frac{1}{100 - 85} = \frac{1}{15}$$

So,

$$Q_{10}^{(1,2)} = \frac{29}{30} \cdot \frac{1}{15} = \frac{29}{450}$$

3. From Theorem 1.18, for the homogeneous Markov Chain,

$$_{\mathbf{k}}\mathbf{Q} = \mathbf{Q}^k$$

Since we begin in State 2, the state probability vector for time $t = 0$ is

$$(0, 1)$$

and we need to examine $(0, 1) \cdot \mathbf{Q}^3$

$$(0, 1) \cdot \mathbf{Q} = (0, 1) \cdot \left\| \begin{array}{cc} 0.4 & 0.6 \\ 0.8 & 0.2 \end{array} \right\| = (0.8, 0.2)$$

$$(0.8, 0.2) \cdot \left\| \begin{array}{cc} 0.4 & 0.6 \\ 0.8 & 0.2 \end{array} \right\| = (0.48, 0.52)$$

$$(0.48, 0.52) \cdot \left\| \begin{array}{cc} 0.4 & 0.6 \\ 0.8 & 0.2 \end{array} \right\| = (0.608, 0.392)$$

Now we can see that the probability of being in state 1 after 3 time periods is 0.608.

4.

$$\mathbf{Q} = \left\| \begin{array}{ccc} 0.6 & 0.3 & 0.1 \\ 0.3 & 0.5 & 0.2 \\ 0 & 0.4 & 0.6 \end{array} \right\|$$

Let Preferred, Standard, and Substandard be States 1, 2, and 3, respectively. Then we begin in State 2 and we want to know the probability of being in state 2 after 3 time periods:

$$(0, 1, 0) \cdot \left\| \begin{array}{ccc} 0.6 & 0.3 & 0.1 \\ 0.3 & 0.5 & 0.2 \\ 0 & 0.4 & 0.6 \end{array} \right\| = (0.3, 0.5, 0.2)$$

$$(0.3, 0.5, 0.2) \cdot \left\| \begin{array}{ccc} 0.6 & 0.3 & 0.1 \\ 0.3 & 0.5 & 0.2 \\ 0 & 0.4 & 0.6 \end{array} \right\| = (0.33, 0.42, 0.25)$$

$$(0.33, 0.42, 0.25) \cdot \left\| \begin{array}{ccc} 0.6 & 0.3 & 0.1 \\ 0.3 & 0.5 & 0.2 \\ 0 & 0.4 & 0.6 \end{array} \right\| = (0.324, 0.409, 0.267)$$

We can see from this last state probability vector that the probability of being in State 2 after 3 steps is 0.409.

5. This is simply

$$\left[Q^{(2,2)}\right]^3 = 0.5^3 = 0.125$$

6. We know the probability of *not* being gone will be one minus the probability of being gone. It's easier to calculate the probability of being gone (make sure you agree!) so we'll start with that. In the notation of the specified example, we're looking for

$${}_2Q_1^{(1,4)}$$

The state probability vector for time $t = 1$ is $(1, 0, 0, 0$. At time $t = 2$, the state probability vector is

$$(1, 0, 0, 0) \cdot \mathbf{Q_1} = (0.70, 0.15, 0.10, 0.05)$$

The state probability vector after two steps is

$$(1, 0, 0, 0) \cdot \mathbf{Q_2} = (--, --, --, 0.1825)$$

where we only calculated the last entry since that is the only one we need to answer the question. So the probability of being in the Gone state after 2 steps is 0.1825 and the probability of not being gone is

$$1 - 0.1825 = 0.8175$$

7. Since we know the employee is active at time 1,

$$(1, 0, 0, 0) \cdot \mathbf{Q_1} = (0.70, 0.15, 0.10, 0.05)$$

$$(0.70, 0.15, 0.10, 0.05) \cdot \mathbf{Q_2} = (0.450, 0.165, 0.2025, 0.1825)$$

$$(0.450, 0.165, 0.2025, 0.1825) \cdot \mathbf{Q_3} = (--, --, --, 0.3535)$$

where we only calculated the term we needed by multiplying the state probability vector by the last column of $\mathbf{Q_3}$.

8. The subject must have the following locations: State 2 at time 3, unknown at time 4, State 1 at time 5, and State 3 at time 6. So, it could move as follows:

(A)

$$2 \longrightarrow 1 \longrightarrow 1 \longrightarrow 3$$

(B)

$$2 \longrightarrow 2 \longrightarrow 1 \longrightarrow 3$$

(C)

$$2 \longrightarrow 3 \longrightarrow 1 \longrightarrow 3$$

(D)

$$2 \longrightarrow 4 \longrightarrow 1 \longrightarrow 3$$

The probabilities of following the paths are as follows:

(A)

$$(0.2)(0.4)(0.3) = 0.024$$

(B)

$$(0.3)(0.1)(0.3) = 0.009$$

(C)

$$(0.35)(0)(0) = 0$$

(D)

$$(0.15)(0)(0) = 0$$

Summing these together gives us our answer of 0.033. (Note that this could also be done by finding the probability the probability the subject is in State 1 at time $t = 5$ and then multiplying by the transition probability from State 1 to State 3 in the next step.

CHAPTER 2 – CASH FLOWS AND THEIR ACTUARIAL PRESENT VALUES

Section 2.2

1. The trick to this problem is to properly read the matrices to pick up the correct cash flows at the correct time period. For this problem, you should have:

Period	*State*	*CashFlow*	*DiscountFactor*
4	1	−−	−−
5	2	42	1.05^{-1}
6	2	56	1.05^{-2}
7	1	65	1.05^{-3}
8	3	73	1.05^{-4}
9	4	83	1.05^{-5}

Applying the interest discounts to the cash flows and summing them results in 272.033.

2. Again, getting the info in the problem straight is the key – the calculations are pretty straightforward. The potential cash flows can be obtained by moving along the following paths:

- State $2 \rightarrow 1$ with a cash flow of 4 (amount equal to the time period), probability of (0.8), and int. discount of 1.25^{-1}

- Similarly, State 2 $\to$ 1 $\to$ 2 $\to$ 1 with CF = 6, prob = (0.8)(0.6)(0.8), int. disc = 1.25^{-3}
- State 2 $\to$ 2 $\to$ 1 with CF = 5, prob = (0.2)(0.8), int. disc = 1.25^{-2}
- State 2 $\to$ 2 $\to$ 2 $\to$ 1 with CF = 6, prob = (0.2)(0.2)(0.8), int. disc = 1.25^{-3}

Multiplying through and summing the 4 values gives 4.3500.

3. Same as problem 2, but with different interest discount factors:

   - State 2 $\to$ 1 with a cash flow of 4 (amount equal to the time period), probability of (0.8), and int. discount of 1.10^{-1}
   - State 2 $\to$ 1 $\to$ 2 $\to$ 1 with CF = 6, prob = (0.8)(0.6)(0.8), int. disc = $\frac{1}{(1.10)(1.15)(1.20)}$
   - State 2 $\to$ 2 $\to$ 1 with CF = 5, prob = (0.2)(0.8), int. disc = $\frac{1}{(1.10)(1.15)}$
   - State 2 $\to$ 2 $\to$ 2 $\to$ 1 with CF = 6, prob = (0.2)(0.2)(0.8), int. disc = $\frac{1}{(1.10)(1.15)(1.20)}$

   Multiplying through and summing the 4 values gives 5.1858.

4. There are three different cash flows, probabilities, and interest factors as follows:

   - State I $\to$ I $\to$ T $\to$ P with a cash flow of 77, probability of (0.3)(0.2)(0.3), and int. discount of 1.25^{-3}.
   - State I $\to$ T $\to$ P with CF = 67, prob = (0.2)(0.4), int. disc = $\frac{1}{(1.25)^2}$
   - State I $\to$ T $\to$ T $\to$ P with a cash flow of 77, probability of (0.2)(0.1)(0.3), and int. discount of 1.25^{-3}.

   Multiplying through and summing the 3 values gives 4.3766.

5. Same cash flows and probabilities, with different interest factors as shown below. Note that, as as example, the interest discount rate for time period 8 back to time period 5 is

   $$(0.05)|7-4| = 0.15$$

   The interest rates for the other periods are calculated similarly.

   - State I $\to$ I $\to$ T $\to$ P with a cash flow of 77, probability of (0.3)(0.2)(0.3), and int. discount of $\frac{1}{(1.05)(1.10)(1.15)}$.
   - State I $\to$ T $\to$ P with CF = 67, prob = (0.2)(0.4), int. disc = $\frac{1}{(1.05)(1.10)}$
   - State I $\to$ T $\to$ T $\to$ P with a cash flow of 77, probability of (0.2)(0.1)(0.3), and int. discount of $((1.05)(1.10)(1.15))^{-1}$.

   Multiplying through and summing the 3 values gives 6.0320.

Section 2.3

1.

Period	*State*	*CashFlow*	*DiscountFactor*
4	1	10	−−
5	2	−−	−−
6	2	−−	−−
7	1	10	1.05^{-3}
8	3	30	1.05^{-4}
9	4	−−	−−

Applying the interest discounts to the cash flows and summing them results in 43.319.

2. The Cash flows are as follows:

   - A cash flow at the start (time 4) of 1
   - State $2 \to 1 \to 2$ with a cash flow of 1, probability of (0.8)(0.6), and int. discount of 1.25^{-2}
   - Similarly, State $2 \to 2$ with CF = 1, prob = (0.2), int. disc = 1.25^{-1}
   - State $2 \to 2 \to 2$ with CF = 1, prob = (0.2)(0.2), int. disc = 1.25^{-2}

   Multiplying through and summing the 4 values gives 1.4928.

3. The cash flows are as follows:

   - A cash flow at the start (time 4) of 1
   - State $2 \to 1 \to 2$ with a cash flow of 1, probability of (0.8)(0.6), and int. discount of $\frac{1}{(1.10)(1.15)}$
   - Similarly, State $2 \to 2$ with CF = 1, prob = (0.2), int. disc = $\frac{1}{(1.10)}$
   - State $2 \to 2 \to 2$ with CF = 1, prob = (0.2)(0.2), int. disc = $\frac{1}{(1.10)(1.15)}$

   Multiplying through and summing the 4 values gives 1.5929.

4. There are seven different cash flows, probabilities, and interest factors as follows:

   - State $I \to I \to I \to T$ with a cash flow of 1, probability of (0.3)(0.2)(0.1), and int. discount of 1.25^{-3}.
   - State $I \to I \to T$ with CF = 1, prob = (0.3)(0.2), int. disc = $\frac{1}{(1.25)^2}$
   - State $I \to I \to T \to T$ with CF = 1, prob = (0.3)(0.2)(0.05), int. disc = $\frac{1}{(1.25)^3}$
   - State $I \to T$ with CF = 1, prob = (0.2), int. disc = $\frac{1}{(1.25)^1}$
   - State $I \to T \to I \to T$ with CF = 1, prob = (0.2)(0.1)(0.1), int. disc = $\frac{1}{(1.25)^3}$
   - State $I \to T \to T$ with CF = 1, prob = (0.2)(0.1), int. disc = $\frac{1}{(1.25)^2}$

- State I $\rightarrow$ T $\rightarrow$ T $\rightarrow$ T with CF = 1, prob = (0.2)(0.1)(0.05), int. disc = $\frac{1}{(1.25)^3}$

Multiplying through and summing the 7 values gives 0.21734.

5. Once more, with feeling – and different interest rates:

- State I $\rightarrow$ I $\rightarrow$ I $\rightarrow$ T with a cash flow of 1, probability of (0.3)(0.2)(0.1), and int. discount of $\frac{1}{(1.05)(1.10)(1.15)}$.
- State I $\rightarrow$ I $\rightarrow$ T with CF = 1, prob = (0.3)(0.2), int. disc = $\frac{1}{(1.05)(1.10)}$
- State I $\rightarrow$ I $\rightarrow$ T $\rightarrow$ T with CF = 1, prob = (0.3)(0.2)(0.05), int. disc = $\frac{1}{(1.05)(1.10)(1.15)}$
- State I $\rightarrow$ T with CF = 1, prob = (0.2), int. disc = $\frac{1}{(1.05)}$
- State I $\rightarrow$ T $\rightarrow$ I $\rightarrow$ T with CF = 1, prob = (0.2)(0.1)(0.1), int. disc = $\frac{1}{(1.05)(1.10)(1.15)}$
- State I $\rightarrow$ T $\rightarrow$ T with CF = 1, prob = (0.2)(0.1), int. disc = $\frac{1}{(1.05)(1.10)}$
- State I $\rightarrow$ T $\rightarrow$ T $\rightarrow$ T with CF = 1, prob = (0.2)(0.1)(0.05), int. disc = $\frac{1}{(1.05)(1.10)(1.15)}$

Multiplying through and summing the 7 values gives 0.26877.

6.

Period	Cash Flow	Prob = $0.6^{(\text{Period})}$	Int Disc. = $1.25^{-(\text{Period})}$	Product
0	100	1.000	1.000	100.000
1	100	0.600	0.800	48.000
2	100	0.360	0.640	23.040
3	100	0.216	0.512	11.059
4	100	0.130	0.410	5.308
5	100	0.078	0.328	2.548
6	100	0.047	0.262	1.223
7	100	0.028	0.210	0.587
8	100	0.017	0.168	0.282
9	100	0.010	0.134	0.135
10	100	0.006	0.107	0.065
11	100	0.004	0.086	0.031
12	100	0.002	0.069	0.015
13	100	0.001	0.055	0.007
14	100	0.001	0.044	0.003
15	100	0.000	0.035	0.002
16	100	0.000	0.028	0.001
17	100	0.000	0.023	0.000

As you can see, by the time you get to time period 17, the probability and interest discount factors are so small that their impact going forward is negligible. Summing the Products of the first 17 periods gives the APV = 193.31.

Section 2.4

1. Using the info from Section 2.2 #2, we know that the present value of benefits is 4.3500.

   From Section 2.3 #2, we know the present value of 1 at the specified times if in State 2 is 1.4928. The benefit premium P is calculated as

$$1.4928 \text{ P} = 4.3500 \longrightarrow P = 2.1940$$

2. Just like Problem 1 above, but with different interest rates. Using the info from Section 2.2 #3, we know that the present value of benefits is 5.1858.

   From Section 2.3 #3, we know the present value of 1 at the specified times if in State 2 is 1.5929. The benefit premium P is calculated as

$$1.5929 \text{ P} = 5.1858 \longrightarrow P = 3.2556$$

3. Using the info from Section 2.2 #4, we know that the present value of benefits is 4.3766.

   From Section 2.3 #4, we know the present value of 1 at the specified times and states is 0.21734. The benefit premium P is calculated as

$$0.21734 \text{ P} = 4.3766 \longrightarrow P = 20.1366$$

4. Using the info from Section 2.2 #5, we know that the present value of benefits is 6.0320.

   From Section 2.3 #5, we know the present value of 1 at the specified times and states is 0.26877. The benefit premium P is calculated as

$$0.26877 \text{ P} = 6.0320 \longrightarrow P = 22.4426$$

5. To calculate the benefit reserve (V_x), we need present value of future benefits (PVFB) and future premiums (PVFP).

$$PVFB = \frac{(5)(0.8)}{(1.25)} + \frac{(6)(0.2)(0.8)}{1.25^2} = 3.8144$$

$$PVFP = 2.914 + \frac{(2.914)(0.2)}{(1.25)} = 3.38024$$

where 2.914 is the benefit premium determined in Problem 1. Make sure you understand where the different probabilities used above come from (remember we're starting in State 2 at time 4).

$$V_x = 3.8144 - 3.3802 = 0.4342$$

6. To calculate the benefit reserve (V_x), we need present value of future benefits (PVFB) and future premiums (PVFP).

$$PVFB = \frac{(77)(0.3)(0.2)}{(1.25)^2} = 2.9568$$

$$PVFP = \frac{(20.137)(0.2)}{1.25} + \frac{(20.137)(0.2)(0.1)}{(1.25)^2} + \frac{(20.137)(0.2)(0.05)}{(1.25)^2} = 3.6086$$

where 2.914 is the benefit premium determined in Problem 1. Make sure you understand where the different probabilities used above come from (remember we're starting in State 2 at time 4).

$$V_x = 2.9568 - 3.6086 = -0.6518$$