Thanks for downloading Chapter 3 of our manual!

This is an early release of the first chapter from the Arch Solutions Course M Study Manual for the Fall 2006 SOA actuarial exam. Here are some (unsolicited) comments from readers of earlier editions:

> *-I am a huge fan of **ARCH**. It is by far the best study manual and I am recommending it to all of my friends. I actually bought [several other manuals]. Now I think I wasted a whole set of money on the others since they always end up confusing me and I always have to come back to **ARCH** for clarification.*
>
> *-I start a seminar on Friday, and I never would have been able to finish and understand the material without your study guide.*
>
> *-I want to personally thank both of you for the fantastic and brilliant work that you did on **ARCH**. Seeing as it's not my first time tackling this exam, I've had the chance to use [several other manuals]; however this is by far superior to all of those products. I have and will continue to recommend it to others in my company.*
>
> *-I would first like to say that I am very happy with your manual so far. I feel that I am progressing through the syllabus much faster than I would have without it, and the depth of understanding that I am on my own giving up due to my not using the texts themselves is more than compensated for by the excellent coverage of the important topics in your manual.*

The manual is currently available at the following bookstores:
www.actexmadriver.com
www.actuarialbookstore.com

Updates to the manual and answers to suggested text exercises will continue to be available at www.archactuarial.com.

One hint for using this manual:
Each chapter of this manual contains plenty of examples with solutions. You are likely to benefit a great deal if every time you get to an example, you cover up the solution and attempt to work it. You will get many of them wrong, especially the first time you see them. But the problem-solving experience will be extremely valuable!

Introduction:

We at ARCH understand that your time is valuable. We appreciate the fact that you've decided to invest some of your study time with us, and we believe our manual will help you make the most of your study time.

The ARCH manual is designed and written in such a way as to help you learn the material as efficiently as possible. The material for the course is broken down into different chapters from the textbooks. The chapters are presented using down-to-earth explanations and we point out the critical concepts and formulas you must know to pass.

On the exam, you will not be asked to explain anything. You will be asked to calculate numerical answers. Therefore, much of our explanation of the material is done by way of numerical examples and practice questions. Examples range from very simple ones(to make sure you know the basic concepts), to thought provoking ones (to help you think about what you've learned and really understand it), to exam questions from prior exams (to get you ready for exam day).

We also suggest problems from the texts for you to work. Many of the problems in the text are not transferable to the exam. Some however, provide useful insight and practice so we've made notes of those. Solutions to these suggested problems are available on the Download Samples page at www.archactuarial.com.

A formula summary for each chapter is included. These summaries are intended to serve as a reference as you familiarize yourself with the syllabus material.

Finally, we have included a full length practice exam of new questions. This exam was designed under the oversight of a consultant for a national testing organization. The questions follow the general form of an SOA exam, but are designed to test concepts not explicitly covered in the early exams under the current exam system. Note that this practice exam is designed to be used in conjunction with the prior Course 3 exams that are available for free on the SOA website at www.soa.org under the “Education and Examination” section. Make sure you work all of these exams!

Robin Cunningham Nathan Hardiman

Actuarial Mathematics: Chapter 3

This text forms the heart and soul of the exam syllabus. The basic principles of life insurance (and annuities) are explained throughout the book. You need to have a solid understanding of this material in order to pass the exam. However, you do not need to understand the majority of the underlying theory in this text. The key points that a student must learn from this text are:

KEYPOINTS:

1. Notation – much of this notation is new. While it can be confusing at first, there is some logic to it. **It will help you to remember and understand the many symbols if you regularly translate the notation into words as you read.**

2. Basic ideas – for example, chapter four introduces a variety of types of insurance. You will want to make sure you have an understanding of these different products and their benefit designs. Another key point is that there are many parallels. Again in Chapter 4, the first part of the chapter considers products which pay a benefit immediately upon death. The second part of the chapter considers the same products except that the benefit is paid at the end of the year in which death occurs. It is helpful to realize that you are really learning only one set of products, with a couple of benefit options, rather than two sets of products. These parallels run throughout the text (e.g., continuous vs curtate functions).

3. Learn key formulas – there is no substitute for being able to recall the formula for, say, a net level premium reserve for term insurance. If you can do this for most of the formulas, you will be ready to answer questions quickly. This manual has tools to help you learn these formulas, so don't feel overwhelmed!

To the text!!!

Chapter 3 is all about notation, definitions, and a few basic ideas that are essential to life contingencies. If you can make yourself comfortable with the symbols and methods of Chapter 3, the rest of Actuarial Mathematics will be easier to absorb.

Section 3.2.1 The Survival Function

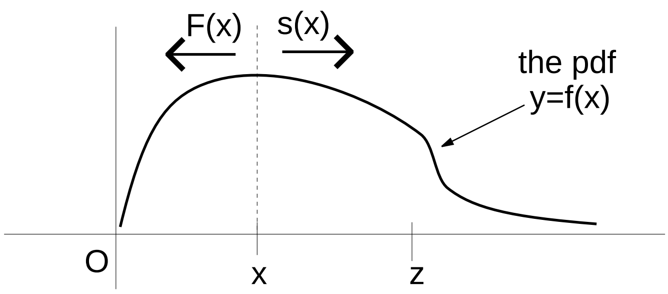
Consider a newborn (i.e. a person whose attained age = 0).

Definitions
X = newborn's age at death
You can also think of X as "the future lifetime of a newborn".

Define $F(x) = \Pr(X \leq x)$, where $x \geq 0$. Read as "the probability that death will occur prior to (or at) age x". In statistics, $F(x)$ is the cumulative distribution function for the future lifetime of a newborn. If $y > x$, it is always true that $F(y) > F(x)$.
This makes sense. For a newborn, $F(98)$, the probability of dying before age 98, is greater than $F(94)$, the probability of dying before age 94.

Define $s(x) = 1 - F(X) = 1 - \Pr(X \leq x)$. The function $s(x)$ is a survival function. Read it as "the probability that death does not occur by age x" or "the probability of attaining (surviving to) age x".

$$\begin{aligned}\Pr(x < X \leq z) &= \text{probability that a newborn dies between ages } x \text{ and } z \\ &= F(z) - F(x) \\ &= [1 - s(z)] - [1 - s(x)] \\ &= s(x) - s(z)\end{aligned}$$



The figure shows the probability distribution function $f(x)$ for death at age x. For any value of x, $F(x)$ is equal to the area **under** the curve $y = f(x)$ and to the left of x. Similarly $s(x)$ is equal to the area under the curve and to the right of x.

By the way, you may have noticed that in our discussion, we dropped the subscript X in $F_X(x)$ ... you can ignore it. I don't know if the authors realize it but they are being a little inconsistent in their treatment of F and s! If two different random variables, say X and Y, referred to the future lifetimes of two different newborns, then you would need to keep the F and s straight for each kid. That's all the subscript is indicating.

Section 3.2.2 Time-until-Death for a Person Age x

Newborns are great, but if our pension and insurance companies are going to make money we need to be able to deal with people who are older than 0. So ...

Consider a person with attained age $= x$.
The simple $F(x)$ and $s(x)$ functions no longer work, since we are now dealing with a person who has already survived to age x. We are facing a conditional probability situation.

$$\Pr(x < X \leq z | X > x)$$

$$= \text{probability that person living at age } x \text{ will die between ages } x \text{ and } z$$

$$= \text{the probability that an } x\text{-year-old will die before turning } z$$

$$= \frac{[F(z) - F(x)]}{[1 - F(x)]}$$

$$= \frac{[s(x) - s(z)]}{[s(x)]}$$

Why is this a conditional probability? Because it is the probability that a newborn will die before age z given that the newborn survives to age x.

EXAMPLE:

1. Write two expressions (one with F only and one with s only) for the probability that a newborn dies between 17 and 40, assuming the newborn dies between 10 and 40.
2. Interpret the following expression in English (or the language of your choice!).

$$\frac{S(20) - S(35)}{1 - S(80)}$$

SOLUTION:

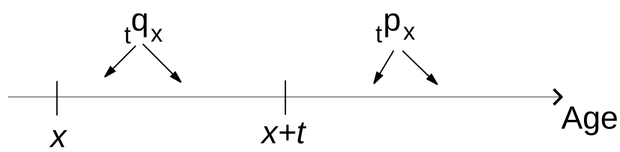
1.
$$\frac{F(40) - F(17)}{F(40) - F(10)} \quad \text{or} \quad \frac{s(17) - s(40)}{s(10) - s(40)}$$

2. "The probability of death between ages 20 and 35, given that the newborn will not attain age 80." ♢

Now, let the symbol "(x)" represent a person age x and let $T(x)$ be the future lifetime of a person age x. (So $T(25)$ is the future lifetime of (25), a twenty-five-year-old.)
Two basic probability functions exist regarding $T(x)$:

$$\begin{aligned} {}_tq_x &= \text{probability that person age } x \text{ will die within } t \text{ years} \\ &= \Pr[T(x) \le t] \text{ where } t \ge 0 \end{aligned}$$

$$\begin{aligned} {}_tp_x &= \text{probability that person age } x \text{ will survive at least } t \text{ years} \\ &= \Pr[T(x) > t] \text{ where } t \ge 0 \end{aligned}$$



In the figure, ${}_tq_x$ is the probability that (x)'s death will occur in the age-interval $(x, x+t)$, and ${}_tp_x$ is the probability that (x)'s death will occur in the age interval $(x+t, \omega)$. (ω represents the oldest possible age to which a person may survive.)

Useful notes:

- ${}_tp_0$ is just $s(t)$.
- If $t = 1$, the convention is to drop the symbol 1, leaving us with either p_x or q_x.

Remember, these are the two basic functions. The formulas that follow are simply take-offs on ${}_tp_x$ or ${}_tq_x$ which you will learn with practice.

The symbol

$${}_{t|u}q_x$$

represents the probability that (x) (that is, a person age x) survives at least t more years, but dies before reaching age $x + t + u$. This is equal to each of the following expressions, each of which you want to be able to put into words:

$$\Pr[t < T(x) \le t + u]$$

$${}_{t+u}q_x - {}_tq_x$$

$${}_tp_x - {}_{t+u}p_x$$

(As with q_x and p_x, if $u = 1$, we drop it, leaving ${}_{t|}q_x$, the probability that (x) will survive t years but not $t + 1$ years.)

Useful formulas:

$$_tp_x = \frac{_{x+t}p_0}{_xp_0} = \frac{s(x+t)}{s(x)}$$

$$_tq_x = 1 - \frac{s(x+t)}{s(x)}$$

$$\begin{aligned} _{t|u}q_x &= \frac{s(x+t) - s(x+t+u)}{s(x)} \\ &= \frac{s(x+t)}{s(x)} * \frac{s(x+t) - s(x+t+u)}{s(x+t)} \\ &= {_tp_x} * {_uq_{x+t}} \end{aligned}$$

This last equation makes sense. It says "The probability of (x) dying between t and $t+u$ years from now ($_{t|u}q_x$) is equal to the probability that (x) will first survive t years ($_tp_x$) and then die within u years ($_uq_{x+t}$)."

If you don't remember anything else from the above, remember the following!

$$_tp_x = \frac{s(x+t)}{s(x)}$$

CONCEPT REVIEW:

1. Write the symbol for the probability that (52) lives to at least age 77.
2. Write the symbol for the probability that a person age 74 dies before age 91.
3. Write the symbol for probability that (33) dies before age 34.
4. Write the symbol for probability that a person age 43 lives to age 50, but doesn't survive to age 67.
5. Write $_{5|6}q_x$ in terms of F and then in terms of p.

SOLUTIONS:

1. $_{25}p_{52}$ 2. $_{17}q_{74}$ 3. q_{33} 4. $_{7|17}q_{43}$

5. $$_{5|6}q_x = \frac{s(x+5) - s(x+11)}{s(x)} = \frac{F(x+11) - F(x+5)}{1 - F(x)}$$

$$= {_5p_x}(1 - {_6p_{x+5}}) \text{ or } = {_5p_x} - {_{11}p_x}.$$ ◇

Remember, it is critical to be able to 'translate' between symbols and words (and vice versa). This not only aids in your understanding of the symbol, but also will help you convert the word problems on the exam into symbols. Then, the symbols will help you to recall appropriate formulas (which you've memorized) to solve the problem at hand.

Section 3.2.3 Curtate-Future-Lifetime

Suppose a person was born on Jan 1, 1900. If they pass away on Sept 30, 1990, how old were they at death? A "continuous" reply would be 90.745 years old. A "curtate" reply would be 90 years old. Curtate is simply a discrete measure – greatest integer – rather than continuous. Your curtate age can be thought of as your age at your last birthday.
(This book and others use 'Curtate' and 'Discrete' interchangably.)

Previously, we defined $T(x)$ to be the future lifetime of (x). This is a continuous function. Now we define

$$\begin{aligned} K(x) &= \text{curtate future lifetime of } (x) \\ &= \text{greatest integer in } T(x) \\ &= \text{number of future years completed by } (x) \text{ prior to death} \\ &= \text{number of future birthdays } (x) \text{ will have the opportunity to celebrate} \end{aligned}$$

A couple of formulas apply:

$$\begin{aligned} \Pr(K(x) = k) &= \Pr(k \le T(x) < k+1) \\ &= \Pr(k < T(x) \le k+1) \\ &= {}_kp_x - {}_{k+1}p_x \\ &= {}_kp_x * q_{x+k} \\ &= {}_{k|}q_x \end{aligned}$$

(Remember, the 1 in front of q has been dropped.)

EXAMPLE:
If $s(x) = \frac{100-x}{100}$ for every x, what is the probability that $K = 19$ for (18)?

SOLUTION:

$$\Pr(K(18) = 19) = {}_{19|}q_{18} = \frac{s(37) - s(38)}{s(18)}$$

$$= \frac{63 - 62}{82} = \frac{1}{82}. \qquad \diamondsuit$$

Section 3.2.4 Force of Mortality

The force of mortality can be thought of as the probability of death at a particular instant given survival up to that time. This is an instantaneous measure, rather than an interval measure. There is good bit of theory in this section, but the most important items are the following formulas and the table of relationships.

$$\mu(x) = \frac{f(x)}{1 - F(x)} = \frac{-s'(x)}{s(x)} \tag{3.2.13}$$

It is very important to know the relationships and requirements given in Table 3.2.1. These will probably be tested on the exam. Below is a summary of the useful information in this table. Each row shows 4 ways to express the function in the left column.

	$F(x)$	$s(x)$	$f(x)$	$\mu(x)$
$F(x)$	$F(x)$	$1 - s(x)$	$\int_0^x f(u)\,du$	$1 - e^{-\int_0^x \mu(t)\,dt}$
$s(x)$	$1 - F(x)$	$s(x)$	$\int_x^\infty f(u)\,du$	$e^{-\int_0^x \mu(t)\,dt}$
$f(x)$	$F'(x)$	$-s'(x)$	$f(x)$	$\mu(x)\,e^{-\int_0^x \mu(t)\,dt}$
$\mu(x)$	$\frac{F'(x)}{1-F(x)}$	$\frac{-s'(x)}{s(x)}$	$\frac{f(x)}{s(x)}$	$\mu(x)$

EXAMPLE: Constant Force of Mortality

If the force of mortality is a constant μ for every age x, show that

1. $s(x) = e^{-\mu x}$ 2. ${}_tp_x = e^{-\mu t}$

SOLUTION:

1.
$$s(x) = e^{-\int_0^x \mu\,dt} = e^{-\mu x}.$$

2.
$$ {}_tp_x = \frac{s(x+t)}{s(x)} = e^{-\mu t}. $$
◇

Sections 3.3-3.5 Life Tables

The Life Table was a foundation of actuarial practice for years. Until recently, 'commutation functions' based on life tables allowed actuaries to perform necessary calculations. Today, computers do this arithmetic, but Life Tables continue to be loaded into valuation systems and used in reporting. For this exam, it is important to understand the definitions and relationships within a life table.

Definitions:

$$\begin{aligned} \ell_0 &= \text{number of people in cohort at age 0, also called the "radix"} \\ \ell_i &= \text{number of people in cohort at age } i \text{ (those remaining from the original } l_0) \\ \omega &= \text{limiting age at which probability of survival} = 0\ (s(x) = 0 \text{ for all } x \geq \omega) \\ {}_n d_x &= \text{number alive at age } x \text{ who die by age } x+n \end{aligned}$$

Relationships:

$$\begin{aligned} \ell_x &= \ell_0 * s(x) \\ q_x &= \frac{\ell_x - \ell_{x+1}}{\ell_x} \\ {}_n q_x &= \frac{\ell_x - \ell_{x+n}}{\ell_x} \\ {}_n p_x &= \frac{\ell_{x+n}}{\ell_x} \\ {}_n d_x &= l_x - l_{x+n} \end{aligned}$$

Illustrative Life Table: Basic Functions

Age	ℓ_x	d_x	1,000 q_x
0	100,000.0	2,042.2	20.4
1	97,957.8	131.6	1.4
2	97,826.3	119.7	1.2
3	97,706.6	109.8	1.1
⋮	⋮	⋮	⋮
40	93,131.6	259.0	2.8
41	92,872.6	276.9	3.0
42	92,595.7	296.5	3.2
43	92,299.2	317.8	3.4

EXAMPLE: Life Table Mortality

Above is an excerpt from the Illustrative Life Table in the book. The following questions are all based on this excerpt.

1. Find $s(42)$.
2. Find ${}_{40}d_2$.
3. Find ${}_{38}q_3$.
4. Find ${}_{2|}q_{40}$.

SOLUTION:

1. $s(42) = \frac{92{,}595.7}{100{,}000} = 0.925957.$
2. ${}_{40}d_2 = \ell_2 - \ell_{42} = 5230.6$
3. ${}_{38}q_3 = 1 - {}_{38}p_3 = 1 - \frac{\ell_{41}}{\ell_3} = 1 - \frac{92{,}872.6}{97{,}706.6} = 0.04947.$
4. ${}_{2|}q_{40} = {}_2p_{40} \cdot q_{42} = \frac{92{,}595.7}{93{,}131.6}(0.0032) = 0.003182.$ ♢

Concepts which follow from the Life Table:
Based on Equation (3.2.13) on an earlier page, we can determine that the probability density function $f(t)$ for $T(x)$ is given by $f(t) = {}_tp_x\mu(x+t)$. This says that the probability that (x) will die at age $x+t$, symbolized by $f(t)$, is equal to the probability that (x) will survive t years and then be hit at that instant by the force of mortality.
Among other things, this tells us that

$$\int_0^\infty {}_tp_x\mu(x+t)dt = \int_0^\infty f(t)dt = 1.$$

The **complete-expectation-of-life** is the expected value of $T(x)$ (or $E[T(x)]$ for fans of Statistics) and is denoted $\overset{\circ}{e}_x$. If you remember how to find the expected value of a continuous random variable, you can figure out that

$$\overset{\circ}{e}_x = E[T(x)] = \int_0^\infty {}_tp_x dt$$

$$\text{Var}[T(x)] = 2\int_0^\infty t \cdot {}_tp_x dt - \overset{\circ}{e}_x^{\,2} \qquad (3.5.4)$$

The book shows how to figure both of these formulas out with integration by parts in Section 3.5.1. I suggest that you memorize these two expressions.

The median future lifetime of (x) is denoted $m(x)$ and simply represents the number m such that ${}_mp_x = {}_mq_x$. In other words, it is the number of years that (x) is equally likely to survive or not survive. It can be found by solving any of the following:

$$\Pr[T(x) > m(x)] = \frac{1}{2}$$

or

$$\frac{s[x+m(x)]}{s(x)} = \frac{1}{2}$$

or

$${}_mp_x = \frac{1}{2}.$$

I think the last equation is the easiest to use and I would forget the others.

The **curtate-expectation-of-life** is $E[K(x)]$ and is denoted e_x (no circle). It is too bad they had to come up with the symbols e_x and $\overset{\circ}{e}_x$ to confuse us. I kept them straight by remembering that "life is a continuous circle" (so the circle means continuous). You might not find this hokey phrase useful to remember but it helped me!

Here are the formulas, note the Continuous/Curtate parallel:

$$e_x = E[K(x)] = \sum_1^{\infty} {}_kp_x$$

$$\mathrm{Var}[K(x)] = \sum_1^{\infty} (2k-1) \cdot {}_kp_x - e_x^2$$

EXAMPLE: Constant Force of Mortality

Find $\overset{\circ}{e}_0$ and $\overset{\circ}{e}_{50}$ if the force of mortality is a constant μ.

SOLUTION:

$$\overset{\circ}{e}_0 = \int_0^{\infty} {}_tp_0 \, dt = \int_0^{\infty} e^{-\mu t} dt = \left[\frac{-1}{\mu} e^{-\mu t}\right]_0^{\infty} = \frac{1}{\mu}$$

$$\overset{\circ}{e}_{50} = \int_0^{\infty} {}_tp_{50} \, dt = \int_0^{\infty} e^{-\mu t} dt = \left[\frac{-1}{\mu} e^{-\mu t}\right]_0^{\infty} = \frac{1}{\mu}$$

This is an important result that you should remember. If the force of mortality is constant, your **future** expected lifetime is the same whether you are 0 (a newborn) or 50. This is not a good model for humans, but works well for things like refrigerators and light bulbs. ♢

EXAMPLE: DeMoivre's Law for Mortality

(We'll be learning about DeMoivre later in this chapter.)

If

$$s(x) = \begin{cases} \frac{50-x}{50} & 0 < x < 50 \\ 0 & \text{Otherwise} \end{cases}$$

for all x between 0 and 50, find e_0 and e_{45}.

SOLUTION:

$$e_0 = \sum_1^{50} {}_tp_0 = \sum_1^{50} \frac{50-t}{50} = 50 - \frac{1}{50} \sum_1^{50} t$$

$$= 50 - \frac{1}{50} \frac{(50)(51)}{2} = 24.5$$

$$e_{45} = \sum_1^{5} {}_tp_{45} = \sum_1^{5} \frac{s(45+t)}{s(45)} = \sum_1^{5} \frac{5-t}{5}$$

$$= \frac{4+3+2+1+0}{5} = 2.$$ ♢

More Life Functions:
The expression L_x denotes the total expected number of years lived between ages x and $x+1$ by survivors of the initial group of ℓ_0 lives. Clearly, those who survive to $x+1$ will live one year between x and $x+1$ and so will contribute one full year to L_x. However, those who die during the year will still contribute a fraction of a year to L_x. This measure takes the fractional years into account along with the full years.

$$L_x = \int_0^1 \ell_{x+t} dt$$

The expression m_x is the **central death rate** over the interval x to $x+1$. Make sure not to confuse m_x with $m(x)$, the median future lifetime!

$$m_x = \frac{(\ell_x - \ell_{x+1})}{L_x}$$

L_x and m_x can be extended to time periods longer than a year, but the idea behind them remains the same.

$$\begin{aligned} {}_nL_x &= \int_0^n \ell_{x+t} dt \\ {}_nm_x &= \frac{\ell_x - \ell_{x+n}}{{}_nL_x} \end{aligned}$$

To finish up these sections of the chapter, there are a few more functions and relationships based on the life table. From the student's perspective, these are somewhat obscure formulas – just the sort likely to show up on the exam!!

Let T_x be the total number of years lived beyond age x by the survivorship group with ℓ_0 initial members (i.e. the l_x people still alive at age x). Be careful with notation. This is not $T(x)$, the future lifetime of (x).

$$T_x = \int_0^\infty \ell_{x+t} dt \tag{3.5.16}$$

Note from the definitions that you can think of T_x as ${}_\infty L_x$.

EXAMPLE: Constant Force of Mortality

If $\ell_0 = 1000$ and the force of mortality is a constant $\mu = 0.1$, find

(A) L_5

(B) m_5

(C) T_5

SOLUTION:

(A)

$$L_5 = \int_0^1 \ell_{5+t}\, dt = \int_0^1 {}_tp_5\, \ell_5.$$

Since

$$\ell_5 = \ell_0 e^{-\mu \cdot 5} = 1000e^{-0.5} = 606.5,$$

we have

$$L_5 = 606.5 \int_0^1 e^{-0.1t}\, dt = 606.5 \left[-10e^{-0.1t}\right]_0^1$$
$$= 606.5 \left[10(1 - e^{-0.1})\right] = 577.16.$$

(B)

$$m_5 = \frac{\ell_5 - \ell_6}{L_5} = \frac{606.5 - 548.8}{577.16} = 0.10$$

This approximates the rate at which people were dying between the 5th and 6th years.

(C)

$$T_5 = \int_0^\infty 1000e^{-0.1(5+t)}\, dt = 606.5 \int_0^\infty e^{-0.1t}\, dt = 6065$$

So if we add up all of the time lived by each of the people alive at $t = 5$, we expect to get a total of 6065 years, or 10 years per person. ♢

Relationship:

$$\frac{T_x}{\ell_x} = \overset{\circ}{e}_x$$

This relationship makes sense. It says that the average number of years lived, $\overset{\circ}{e}_x$, by the members of l_x is equal to the total number of years lived by this group divided by l_x.

We can determine the average number of years lived between x and $x+n$ by the ℓ_x survivors at age x as:

$$\frac{{}_nL_x}{\ell_x} = \int_0^n {}_tp_x dt$$

$$\frac{{}_nL_x}{\ell_x} = \text{n-year temporary complete life expectancy of } (x)$$
$$= \overset{\circ}{e}_{x:\overline{n}|} \quad \text{(p.71)}$$

Let $a(x)$ be the average number of years lived between ages x and $x+1$ by those of the survivorship group who die between those ages. Generally, when using this formula, we assume a uniform distribution of deaths over the interval $(x, x+1)$. With this assumption,

$$a(x) = 1/2.$$

Without this assumption,

$$a(x) = \frac{\int_0^1 t \cdot \ell_{x+t}\, \mu(x+t)dt}{\int_0^1 \ell_{x+t}\, \mu(x+t)dt}$$
$$= \frac{\int_0^1 t \cdot {}_tp_x\, \mu(x+t)dt}{\int_0^1 {}_tp_x\, \mu(x+t)dt}$$

Section 3.5.2 Recursion Formulas

We have included this section since it is in the text, and potentially, a question using this notation could appear on the exam. However, if you are like us, you will find recursion easier to understand when it arises more naturally in later chapters. I (Robin) did not bother to learn the formal recursion stuff at all before taking the exam because I thought my limited time was better spent on other stuff. I wasn't disappointed when I took the exam, but you never know. If you think it is preferable, you might be able to skip this section, or come back to it if time permits.

These are basically ways to avoid working integrals. They are based on the Trapezoid Rule for integration – maybe you remember the trapezoid rule from calculus.

Backward:

$$u(x) = c(x) + d(x) * u(x+1)$$

Forward:

$$u(x+1) = \frac{u(x) - c(x)}{d(x)}$$

Note that the Forward Method is simply an algebraic recombination of the Backward Method. Note also that this Forward formula is different from the book – work out the formulas yourself to convince yourself of their equivalence. Then, learn whichever form you find more straightforward.

The text shows how to use these formulas to compute e_x and $\mathring{e}_x$ starting with e_ω and $\mathring{e}_\omega$ and working backward. For e_x, using the recursion once will produce $e_{\omega-1}$, the second iteration will produce $e_{\omega-2}$, etc. until you get all the way back to e_0, when you will have produced a list of e_x for every x between 0 and ω.

The formulas are: for e_x,

$$u(x) = e_x$$

$$c(x) = p_x$$

$$d(x) = p_x$$

$$\text{Starting Value} = e_\omega = u(\omega) = 0$$

So to start, set $x + 1 = \omega$ and the recursion will produce $u(x) = u(\omega - 1)$.

For $\overset{\circ}{e}_x$,

$$u(x) = \overset{\circ}{e}_x$$
$$c(x) = \int_0^1 {}_sp_x ds$$
$$d(x) = p_x$$
$$\text{Starting Value} = \overset{\circ}{e}_\omega = u(\omega) = 0$$

Section 3.6 Assumptions for Fractional Ages

(OK, you can start paying attention again ....)

The random variable T is a continuous measure of remaining lifetime. The life table has been developed as an approximation of T, using a curtate variable K. As we've discussed, K is only defined at integers. So, we need some way to measure between two integer ages. Three popular methods have been developed.

For all of the methods that follow, let x be an integer and let $0 \leq t \leq 1$. Suppose that we know the value of $s(x)$ for the two integers x and $x+1$ and we want to approximate s at values between x and $x+1$. In other words, we want to approximate $s(x+t)$ where $0 \leq t \leq 1$.

Method 1: Linear Interpolation:

$$s(x+t) = (1-t)s(x) + t \cdot s(x+1)$$

This important method is also known as "Uniform Distribution of Deaths", UDD. In this case, ${}_tp_x$ is assumed to be linear function. This method assumes that the deaths occuring between ages x and $x+1$ are evenly spread out between the two ages. As you might imagine, this is usually not quite correct, but is a pretty good approximation. (Please note: the linearity of ${}_tp_x$ is only assumed to hold up to $t = 1$!)

Method 2: Exponential Interpolation:

$$\log s(x+t) = (1-t)\log s(x) + t \cdot \log s(x+1)$$

This is another important method known as the "Constant force of mortality assumption". Here, ${}_tp_x$ is an exponential function.

Method 3: Harmonic Interpolation:

$$\frac{1}{s(x+t)} = \frac{1-t}{s(x)} + \frac{t}{s(x+1)}$$

This method is less important – as far as I can tell it exists only to make certain formulas work out nicely later on! (e.g. see line 3 of Table 3.6.1 below.) It is known as the "Hyperbolic assumption" or the "Balducci assumption". In this case ${}_tp_x$ is assumed to a hyperbolic curve. Even though this method leads to weird and counterintuitive results (you get less likely to die as you age during the year), it makes regular appearances on the exam, so you must know it.

Function	**Uniform Distribution**	**Constant Force**	**Hyperbolic**
${}_tq_x$	tq_x	$1-p_x^t$	$\frac{tq_x}{1-(1-t)q_x}$
$\mu(x+t)$	$\frac{q_x}{1-tq_x}$	$-\log p_x$	$\frac{q_x}{1-(1-t)q_x}$
${}_{1-t}q_{x+t}$	$\frac{(1-t)q_x}{1-tq_x}$	$1-p_x^{1-t}$	$(1-t)q_x$
${}_yq_{x+t}$	$\frac{yq_x}{1-tq_x}$	$1-p_x^y$	$\frac{yq_x}{1-(1-y-t)q_x}$
${}_tp_x$	$1-tq_x$	p_x^t	$\frac{p_x}{1-(1-t)q_x}$
${}_tp_x\mu(x+t)$	q_x	$-p_x^t\log p_x$	$\frac{q_xp_x}{[1-(1-t)q_x]^2}$

Table 3.6.1

Table 3.6.1 is an excellent summary of the relationships that might be tested. Learning this table can save a great deal of time in the heat of the battle on exam day! If you like deriving expressions, and you really don't like memorization, you could memorize only Lines 5 and 4 of the table (in that order!).

Then ${}_tq_x$ is just $(1-{}_tp_x)$ (easy enough on exam day) – this gives Line 1. Line 3 is quick by substituting $(1-t)$ in for y in Line 4. Finally, you can get lines 2 and 6 by remembering that

$${}_tp_x\mu(x+t) = -\frac{d}{dt}({}_tp_x).$$

This last equation is something you should remember for the test anyway.

A couple of time-saving formulas that are valid **under UDD only!!**

$$\mathring{e}_x = e_x + \frac{1}{2}$$

$$\text{Var}(T) = \text{Var}(K) + \frac{1}{12}$$

EXAMPLE:

You are given that $q_x = 0.1$. Find (A) ${}_{0.5}q_x$, (B) ${}_{0.5}q_{x+0.5}$ under each of

- Uniform Density of Deaths (UDD)
- Exponentially distributed deaths (constant force)
- Harmonic Interpolation (Balducci, hyberbolic)

SOLUTION:

(A)
- UDD:
$${}_{0.5}q_x = (0.5)(0.1) = 0.05$$
- CF:
$${}_{0.5}q_x = 1 - (0.9)^{0.5} = 0.0513$$
- Balducci:
$${}_{0.5}q_x = \frac{(0.5)(0.1)}{1 - (0.5)(0.1)} = 0.0526.$$

(B)
- UDD:
$${}_{0.5}q_{x+0.5} = \frac{(0.5)(0.1)}{1 - (0.5)(0.1)} = 0.0526$$
- CF:
$${}_{0.5}q_{x+0.5} = 1 - (0.9)^{0.5} = 0.0513$$
- Balducci:
$${}_{0.5}q_{x+0.5} = (0.5)(0.1) = 0.05.$$
♢

Notice that with the Balducci (hyperbolic) assumption, if you survive the first half of the year, your mortality actually decreases!

Section 3.7 Some Analytical Laws of Mortality

Although computers have rendered Analytical Laws of Mortality less imperative to our profession, they are still important for understanding mortality and particularly for passing the exam. The book describes four basic analytical laws/formulas:

De Moivre's Law:

$$\mu(x) = (\omega - x)^{-1} \text{ and } s(x) = 1 - \frac{x}{\omega}, \qquad \text{where } 0 \le x < \omega$$

Gompertz' Law:

$$\mu(x) = Bc^x \text{ and } s(x) = \exp[-m(c^x - 1)]$$

$$\text{where } B > 0,\ c > 1,\ m = \frac{B}{\log c},\ x \ge 0$$

Makeham's Law:

$$\mu(x) = A + Bc^x \text{ and } s(x) = \exp[-Ax - m(c^x - 1)]$$

$$\text{where } B > 0,\ A \geq -B,\ c > 1,\ m = \frac{B}{\log c},\ x \geq 0$$

Weibull's Law:

$$\mu(x) = kx^n \text{ and } s(x) = \exp(-ux^{n+1})$$

$$\text{where } k > 0,\ n > 0,\ u = \frac{k}{(n+1)},\ x \geq 0$$

Notes:

- Gompertz is simply Makeham with $A = 0$.
- If $c = 1$ in Gompertz or Makeham, the exponential/constant force of mortality results.
- In Makeham's law, A is the "accident hazard" while Bc^x is the "hazard of aging".

We will be seeing more of these laws later in the book. To be ready for the exam, you should become intimately familiar with De Moivre's Law. DeMoivre's Law says that at age x, you are equally likely to die in any year between x and ω.

Here are some key life-functions for DeMoivre's Law in the form of an example. If you don't read the proofs, still make an effort to understand the formulas and what they mean.

EXAMPLE:

Under DeMoivre's law, show that

1. $${}_tp_x = \frac{\omega - x - t}{\omega - x}$$
2. $${}_tq_x = \frac{t}{\omega - x}$$
3. $$\overset{\circ}{e}_0 = \frac{\omega}{2}$$
4. $$e_0 = \frac{\omega - 1}{2}$$
5. $$\overset{\circ}{e}_x = \frac{\omega - x}{2}$$
6. $$e_x = \frac{\omega - x - 1}{2}$$

SOLUTION:

1. $$ {}_tp_x = \frac{s(x+t)}{s(x)} = \frac{\omega - x - t}{\omega - x} $$

2. $$ {}_tq_x = 1 - {}_tp_x = \frac{t}{\omega - x} $$

3. $$ \overset{\circ}{e}_0 = \int_0^{\omega} \frac{\omega - t}{\omega} \, dt = \left[-\frac{(\omega - t)^2}{2\omega} \right] = \frac{\omega}{2} $$

4. $$ e_0 = \sum_1^{\omega} {}_kp_0 = \sum_1^{\omega} \frac{\omega - k}{\omega} = \omega - \frac{1}{\omega} \sum_1^{\omega} k = \frac{\omega - 1}{2} $$

5. $$ \overset{\circ}{e}_x = \int_0^{\omega - x} \frac{\omega - x - t}{\omega - x} \, dt = \frac{\omega - x}{2} $$

6. Since mortality is uniform over all years under DeMoivre's law, it is uniform over each individual year, so UDD applies. Therefore

$$ e_x = \overset{\circ}{e}_x - \frac{1}{2} = \frac{\omega - x - 1}{2} $$

We could have done (4) this way also. ♢

Modified DeMoivre's Law

Often on the exam, a modified version of DeMoivre's Law arises. This occurs, for example, when

$$ \mu(x) = \frac{c}{\omega - x} $$

where c is a positive constant. This gives rise to a set of formulas for each of the quantities found for standard DeMoivre's Law ($c = 1$) in the example above. All of the formulas for Modified DeMoivre's Law are listed in the formula summary at the end of this chapter.

Section 3.8 Select and Ultimate Tables

Suppose you are trying to issue life insurance policies and two 45 year-old women apply for policies. You want to make sure you charge appropriate premiums for each one to cover the cost of insuring them over time. One of the women is simply picked from the population at large. The second women was picked from a group of women who recently passed a comprehensive physical exam with flying colors – significantly healthier than the general population.

Is it equitable to charge the same premium to the two women? No – because you have additional information about the second woman that would cause you to expect her to have better “mortality experience” than the general population. Thus, to her premiums, you might apply a “select” mortality table that reflects better the mortality experience of very healthy 45 year olds. However, after 15 years, research might show that being very healthy at 45 does not indicate much of anything about health at age 60. So, you might want to go back to using standard mortality rates at age 60 regardless of status at age 45. After all, 15 years is plenty of time to take up smoking, eat lots of fried foods, etc.

This simple scenario illustrates the idea behind select and ultimate tables. For some period of time, you expect mortality to be different than that for the general population – the “select period”. However, at some point, you’re just not sure of this special status anymore, so those folks fall back into the pack at some point – the “ultimate” table.

Consider table 3.8.1. The symbol $[x]$ signifies an x-yr old with “select” status. Note that for the first two years (columns 1,2), select mortality applies with $q_{[x]}$ and $q_{[x]+1}$. However, at duration 3 (column 3), it’s back to standard mortality, q_{x+2}. This table assumes the “selection effect” wears off in just 2 years.

Excerpt from the AF80 Select-and-Ultimate Table in Bowers, et al.

	(1)	**(2)**	**(3)**	**(4)**	**(5)**	**(6)**	**(7)**
$[x]$	1,000 $q_{[x]}$	1,000 $q_{[x]+1}$	1,000 q_{x+2}	$\ell_{[x]}$	$\ell_{[x]+1}$	ℓ_{x+2}	$x+2$
30	0.222	0.330	0.422	9,906.74	9,904.54	9,901.27	32
31	0.234	0.352	0.459	9,902.89	9,900.58	9,897.09	33
32	0.250	0.377	0.500	9,898.75	9,896.28	9,892.55	34
33	0.269	0.407	0.545	9,894.29	9,891.63	9,887.60	35
34	0.291	0.441	0.596	9,889.45	9,886.57	9,882.21	36

Table 3.8.1

Here are a few useful formulas and relationships. In general,

$$q_{[x]} < q_{[x-1]+1} < q_x$$

Why would this hold? The expression q_x represents a pick from the general population. The expression $q_{[x]}$ indicates special knowledge about the situation – for example, recently passing a physical exam (This formula assumes we are trying to select out healthy people, of course). The expression $q_{[x-1]+1}$ indicates special knowledge about the situation as before – for example, recently passing a physical exam, but this time the applicant has had a year for health to deteriorate since she was examined at age $x-1$ (one year ago) rather than at age x.

EXAMPLE: Select and Ultimate Life Table

Using the select and ultimate life table shown above find the value of

$$1000\left({}_3q_{32} - {}_3q_{[32]}\right).$$

SOLUTION:

$_3q_{32}$ deals only with the ultimate table so I am only interested in the values of $l_x + 2$ in Column (6).

$$_3q_{32} = 1 - {_3p_{32}} = 1 - \frac{\ell_{35}}{\ell_{32}} = 1 - \frac{9{,}888}{9{,}901} = 0.001313$$

For $_3q_{[32]}$, we need $\ell_{[32]}$ and $\ell_{[32]+3}$ which is just ℓ_{35} since the select period is only 2 years. So

$$_3q_{[32]} = 1 - \frac{\ell_{35}}{\ell_{[32]}} = 1 - \frac{9{,}888}{9{,}899} = 0.001111$$

So the answer is 0.202. ♢

Two important points regarding this example:

- The probability that [32] will die in the next 3 years is lower if [32] is taken from a select group. People you are sure are healthy should be less likely to die than someone drawn from the general popluation.
- To follow the people alive from the 9898.75 'selected' at age 32, you first follow the numbers to the right until you hit the ultimate column and then proceed down the ultimate column. This is useful! You can quickly evaluate that

$$_5p_{[31]} = \frac{9882}{9903}$$

by counting off 5 years – 2 to the right and then 3 down.

Conclusion

Chap. 3 introduces a lot of new concepts and notation. Make sure you understand the notation in Table 3.9.1 – this is the foundation for the rest of the text.

Chapter 3 Suggested Problems: 1(do first row last), 5, 6, 7, 9, 12, 18abc, 20, 28, 30, 36, 39 There are lots for this chapter, some chapters in this book will have very few. (Solutions are available at archactuarial.com on the Download Samples page.)

Chapter 3 Formula Summary:

$$s(x) = 1 - F(X) = 1 - \Pr(X \leq x)$$
$$\Pr(x < X \leq z) = F(z) - F(x) = s(x) - s(z)$$
$$\Pr(x < X \leq z | X > x) = \frac{[F(z) - F(x)]}{[1 - F(x)]} = \frac{[s(x) - s(z)]}{[s(x)]}$$

$${}_tp_x = e^{\left[-\int_0^t \mu(x+s)ds\right]} \qquad {}_tp_x = \frac{s(x+t)}{s(x)} \qquad {}_tq_x = 1 - {}_tp_x$$

$${}_{t|u}q_x = {}_tp_x * {}_uq_{x+t} \qquad {}_{t|u}q_x = {}_{t+u}q_x - {}_tq_x \qquad {}_{t|u}q_x = {}_tp_x - {}_{t+u}p_x$$

$$\begin{aligned} \Pr(K(x) = k) &= {}_kp_x - {}_{k+1}p_x \\ &= {}_kp_x * q_{x+k} \\ &= {}_{k|}q_x \end{aligned}$$

Life Tables:

$$\ell_x = \ell_0 * s(x) \qquad {}_nd_x = l_x - l_{x+n}$$

$$q_x = \frac{\ell_x - \ell_{x+1}}{\ell_x} \qquad p_x = \frac{\ell_{x+1}}{\ell_x}$$

$${}_nq_x = \frac{\ell_x - \ell_{x+n}}{\ell_x} \qquad {}_np_x = \frac{\ell_{x+n}}{\ell_x}$$

Constant Force of Mortality
If the force of mortality is a constant μ for every age x,

$$s(x) = e^{-\mu x} \qquad {}_tp_x = e^{-\mu t}$$

$$\overset{\circ}{e}_x = \frac{1}{\mu} \qquad \text{Var}[T] = \frac{1}{\mu^2}$$

Expected Future Lifetime

$\overset{\circ}{e}_x = E[T(x)] = \int_0^\infty {}_tp_x dt$ $\quad$ $\text{Var}[T(x)] = 2\int_0^\infty t \cdot {}_tp_x dt - \overset{\circ}{e}_x^2$

$e_x = E[K(x)] = \sum_1^\infty {}_kp_x$ $\quad$ $\text{Var}[K(x)] = \sum_1^\infty (2k-1) \cdot {}_kp_x - e_x^2$

Under UDD only,

$\overset{\circ}{e}_x = e_x + \frac{1}{2}$ $\quad$ $\text{Var}(T) = \text{Var}(K) + \frac{1}{12}$

Median future lifetime

$$\Pr[T(x) > m(x)] = \frac{1}{2} \qquad \frac{s[x + m(x)]}{s(x)} = \frac{1}{2} \qquad {}_mp_x = \frac{1}{2}$$

Make sure not to confuse m_x with $m(x)$, the median future lifetime!

$L_x = \int_0^1 \ell_{x+t} dt$ $\quad$ ${}_nL_x = \int_0^n \ell_{x+t} dt$

$m_x = \frac{(\ell_x - \ell_{x+1})}{L_x}$ $\quad$ ${}_nm_x = \frac{\ell_x - \ell_{x+n}}{{}_nL_x}$

$T_x = \int_0^\infty \ell_{x+t} dt$

$\frac{T_x}{\ell_x} = \overset{\circ}{e}_x$ $\quad$ $\frac{{}_nL_x}{\ell_x} = \int_0^n {}_tp_x dt = \overset{\circ}{e}_{x:\overline{n}|}$

With the assumption of uniform distribution of deaths over the interval $(x, x+1)$,

$$a(x) = 1/2.$$

Without this assumption,

$$a(x) = \frac{\int_0^1 t \cdot \ell_{x+t}\, \mu(x+t) dt}{\int_0^1 \ell_{x+t}\, \mu(x+t) dt} = \frac{\int_0^1 t \cdot {}_tp_x\, \mu(x+t) dt}{\int_0^1 {}_tp_x\, \mu(x+t) dt}$$

De Moivre's Law: $\mu(x) = (\omega - x)^{-1}$ and $s(x) = 1 - \dfrac{x}{\omega}$, where $0 \le x < \omega$

Gompertz' Law: $\mu(x) = Bc^x$ and $s(x) = \exp[-m(c^x - 1)]$

$$\text{where } B > 0,\ c > 1,\ m = \frac{B}{\log c},\ x \ge 0$$

Makeham's Law: $\mu(x) = A + Bc^x$ and $s(x) = \exp[-Ax - m(c^x - 1)]$

$$\text{where } B > 0,\ A \ge -B,\ c > 1,\ m = \frac{B}{\log c},\ x \ge 0$$

Weibull's Law: $\mu(x) = kx^n$ and $s(x) = \exp(-ux^{n+1})$

$$\text{where } k > 0,\ n > 0,\ u = \frac{k}{(n+1)},\ x \ge 0$$

DeMoivre's Law and Modified DeMoivre's Law:
If x is subject to DeMoivre's Law with maximum age ω, then all of the relations on the left below are true. The relations on the right are for Modified DeMoivre's Law with $c > 0$.

DeMoivre	Modified DeMoivre
$\mu(x) = \frac{1}{\omega-x}$	$\mu(x) = \frac{c}{\omega-x}$
$s(x) = \frac{\omega-x}{\omega}$	$s(x) = \left(\frac{\omega-x}{\omega}\right)^c$
$\ell_x = \ell_0 \cdot \frac{\omega-x}{\omega}$	$\ell_0 \left(\frac{\omega-x}{\omega}\right)^c$
$\mathring{e}_x = \mathrm{E}[T] = \frac{\omega-x}{2}$	$\mathring{e}_x = \frac{\omega-x}{c+1}$
$\mathrm{Var}[T] = \frac{(\omega-x)^2}{12}$	$\mathrm{Var}[T] = \frac{(\omega-x)^2\,c}{(c+1)^2(c+2)}$
${}_tp_x = \frac{\omega-x-t}{\omega-x}$	${}_tp_x = \left(\frac{\omega-x-t}{\omega-x}\right)^c$
$\mu_x(t) = \frac{1}{\omega-x-t}$	$\mu_x(t) = \frac{c}{\omega-x-t}$

Be careful on the exam - Modified DeMoivre problems are often disguised in a question that starts something like

> You are given
>
> - $s(x) = (1 - \frac{x}{80})^2$
> - $\cdots$

In cases like this, you have to recognize that the question is just a modified DeMoivre written in a different algebraic form.

Note that in this table, the function listed at left are given in terms of the functions across the top row!

	$F(x)$	$s(x)$	$f(x)$	$\mu(x)$
$F(x)$	$F(x)$	$1-s(x)$	$\int_0^x f(u)\,du$	$1-e^{-\int_0^x \mu(t)\,dt}$
$s(x)$	$1-F(x)$	$s(x)$	$\int_x^\infty f(u)\,du$	$e^{-\int_0^x \mu(t)\,dt}$
$f(x)$	$F'(x)$	$-s'(x)$	$f(x)$	$\mu(x)\,e^{-\int_0^x \mu(t)\,dt}$
$\mu(x)$	$\frac{F'(x)}{1-F(x)}$	$\frac{-s'(x)}{s(x)}$	$\frac{f(x)}{s(x)}$	$\mu(x)$

Assumptions for fractional ages.

Function	**Uniform Distribution**	**Constant Force**	**Hyperbolic**
${}_tq_x$	tq_x	$1-p_x^t$	$\frac{tq_x}{1-(1-t)q_x}$
$\mu(x+t)$	$\frac{q_x}{1-tq_x}$	$-\log p_x$	$\frac{q_x}{1-(1-t)q_x}$
${}_{1-t}q_{x+t}$	$\frac{(1-t)q_x}{1-tq_x}$	$1-p_x^{1-t}$	$(1-t)q_x$
${}_yq_{x+t}$	$\frac{yq_x}{1-tq_x}$	$1-p_x^y$	$\frac{yq_x}{1-(1-y-t)q_x}$
${}_tp_x$	$1-tq_x$	p_x^t	$\frac{p_x}{1-(1-t)q_x}$
${}_tp_x\mu(x+t)$	q_x	$-p_x^t\log p_x$	$\frac{q_xp_x}{[1-(1-t)q_x]^2}$

Past SOA/CAS Exam Questions:

All of these questions have appeared on SOA/CAS exams between the years 2000 and 2005. You will find that they often involve some 'clever' thinking in addition to knowledge of actuarial math. These questions are used with permission.

1. Given:

   (i) $\overset{\circ}{e}_0 = 25$

   (ii) $l_x = \omega - x, \quad 0 \le x \le \omega$

   (iii) $T(x)$ is the future lifetime random variable.

   Calculate Var[$T(10)$].

   (A) 65 (B)93 (C) 133 (D) 178 (E) 333

   Solution:

$$\overset{\circ}{e}_0 = \int_0^{\omega} \left(1 - \frac{t}{\omega}\right) dt = \omega - \frac{\omega^2}{2\omega} = \frac{\omega}{2} = 25 \Rightarrow \omega = 50$$

$$\overset{\circ}{e}_{10} = \int_0^{40} \left(1 - \frac{t}{40}\right) dt = 40 - \frac{40^2}{(2)(40)} = 20$$

$$\text{Var}\left[T(x)\right] = 2\int_0^{40} t\left(1 - \frac{t}{40}\right) dt - (20)^2 = 2\left[\frac{t^2}{2} - \frac{t^3}{3 \cdot 40}\right]_0^{40} - (20)^2 = 133$$

   Key: C

2. For a certain mortality table, you are given:

   (i) $\mu(80.5) = 0.0202$

   (ii) $\mu(81.5) = 0.0408$

   (iii) $\mu(82.5) = 0.0619$

   (iv) Deaths are uniformly distributed between integral ages.

   Calculate the probability that a person age 80.5 will die within two years.

   (A) 0.0782 (B) 0.0785 (C) 0.0790 (D) 0.0796 (E) 0.0800

Solution:

$$0.0408 = \mu(81.5) = \frac{q_{81}}{1-(1/2)q_{81}} \Rightarrow q_{81} = 0.0400$$

Similiarly,

$$q_{80} = 0.0200 \text{ and } q_{82} = 0.0600$$

$$ {}_2q_{80.5} = {}_{1/2}q_{80.5} + {}_{1/2}p_{80.5}\left[q_{81} + p_{81}\cdot {}_{1/2}q_{82}\right]$$

$$= \frac{0.01}{0.99} + \frac{0.98}{0.99}[0.04 + 0.96(0.03)] = 0.0782$$

Key: A

3. Mortality for Audra, age 25, follows De Moivre's law with $\omega = 100$. If she takes up hot air ballooning for the coming year, her assumed mortality will be adjusted so that for the coming year only, she will have a constant force of mortality of 0.1.

   Calculate the decrease in the 11-year temporary complete life expectancy for Audra if she takes up hot air ballooning.

   (A) 0.10 (B) 0.35 (C) 0.60 (D) 0.80 (E) 1.00

Solution: STANDARD:

$$\mathring{e}_{25:\overline{11}|} = \int_0^{11}(1-\frac{t}{75})dt = t - \frac{t^2}{2\times 75}\,|_0^{11} = 10.1933$$

MODIFIED:

$$p_{25} = e^{-\int_0^1 0.1ds} = e^{-0.1} = 0.90484$$

$$\begin{aligned}
\mathring{e}_{25:\overline{11}|} &= \int_0^1 {}_tp_{25}dt + p_{25}\int_0^{10}(1-\frac{t}{74})dt \\
&= \int_0^1 e^{-0.1t}dt + e^{-0.1}\int_0^{10}(1-\frac{t}{74})dt \\
&= \frac{1-e^{-0.1}}{0.1} + e^{-0.1}\left(t - \frac{t^2}{2\times 74}\right)|_0^{10} \\
&= 0.95163 + 0.90484(9.32432) = 9.3886
\end{aligned}$$

The difference is 0.8047. **Key: D**

4. You are given the following extract from a select-and-ultimate mortality table with a 2-year select period:

x	$\ell_{[x]}$	$\ell_{[x]+1}$	ℓ_{x+2}	$x+2$
60	80,625	79,954	78,839	62
61	79,137	78,402	77,252	63
62	77,575	76,770	75,578	64

Assume that deaths are uniformly distributed between integral ages.

Calculate ${}_{0.9}q_{[60]+0.6}$.

(A) 0.0102 (B) 0.0103 (C) 0.0104 (D) 0.0105 (E) 0.0106

Solution:

$$\ell_{[60]+0.6} = (0.6)(79{,}954) + (0.4)(80{,}625) = 80{,}222.4$$

$$\ell_{[60]+1.5} = (0.5)(79{,}954) + (0.5)(78{,}839) = 79{,}396.5$$

$${}_{0.9}q_{[60]+0.6} = \frac{80{,}222.4 - 79{,}396.5}{80{,}222.4} = 0.0103$$

Key: B

5. Given:

(i) $\mu(x) = F + e^{2x},\ x \geq 0$

(ii) ${}_{0.4}p_0 = 0.50$

Calculate F.

(A) -0.20 (B) -0.09 (C) 0.00 (D) 0.09 (E) 0.20

Solution:

$$ {}_{0.4}p_0 = 0.5 = e^{-\int_0^{0.4}\left(F+e^{2x}\right)dx} $$

$$ = e^{-0.4F-\left[\frac{e^{2x}}{2}\right]_0^{0.4}} = e^{-0.4F-\left(\frac{e^{0.8}-1}{2}\right)} $$

$$ \Rightarrow 0.5 = e^{-0.4F-0.6128} \qquad \Rightarrow \ln(0.5) = -0.4F - 0.6128 $$

$$ \Rightarrow -0.6931 = -0.4F - 0.6128 \qquad \Rightarrow F = 0.20 $$

Key: E

6. An actuary is modeling the mortality of a group of 1000 people, each age 95, for the next three years.

   The actuary starts by calculating the expected number of survivors at each integral age by

   $$ \ell_{95+k} = 1000\,{}_kp_{95}, \;\; k = 1, 2, 3 $$

   The actuary subsequently calculates the expected number of survivors at the middle of each year using the assumption that deaths are uniformly distributed over each year of age.

   This is the result of the actuary's model:

Age	Survivors
95	1000
95.5	800
96	600
96.5	480
97	−−
97.5	288
98	−−

   The actuary decides to change his assumption for mortality at fractional ages to the constant force assumption. He retains his original assumption for each ${}_kp_{95}$.

   Calculate the revised expected number of survivors at age 97.5.

   (A) 270 (B) 273 (C) 276 (D) 279 (E) 282

Solution: From UDD, $l_{96.5} = \dfrac{l_{96} + l_{97}}{2}$.

$$480 = \frac{600 + l_{97}}{2} \longrightarrow l_{97} = 360$$

Likewise, from $l_{97} = 360$ and $l_{97.5} = 288$, we get $l_{98} = 216$.

For constant force, $e^{-\mu} = \dfrac{l_{98}}{l_{97}} = \dfrac{216}{360} = 0.6$

$${}_{0.5}p_x = e^{-0.5\mu} = (0.6)^{\frac{1}{2}} = 0.7746$$

$$l_{97.5} = (0.7746)l_{97} = (0.7746)(360) = 278.86$$

Key: D

7. For a 4-year college, you are given the following probabilities for dropout from all causes:

- $q_0 = 0.15$
- $q_1 = 0.10$
- $q_2 = 0.05$
- $q_3 = 0.01$

Dropouts are uniformly distributed over each year.

Compute the temporary 1.5-year complete expected college lifetime of a student entering the second year, $\mathring{e}_{1:\overline{1.5}|}$.

(A) 1.25 (B) 1.3 (C) 1.35 (D) 1.4 (E) 1.45

Solution:

$$\begin{aligned}
\mathring{e}_{1:\overline{1.5}|} &= \int_0^{1.5} {}_tp_1 dt \\
&= \int_0^1 {}_tp_1 dt + {}_1p_1 \int_0^{0.5} {}_xp_2 dx \\
&= \int_0^1 (1 - 0.1t)dt + 0.9 \int_0^{0.5} (1 - 0.05x)dx \\
&= \left[t - \frac{0.1t^2}{2}\right]_0^1 + 0.9\left[x - \frac{0.05x^2}{2}\right]_0^{0.5} \\
&= 0.95 + 0.444 = 1.394
\end{aligned}$$

Key: D

8. For a given life age 30, it is estimated that an impact of a medical breakthrough will be an increase of 4 years in $\mathring{e}_{30}$, the complete expectation of life.

   Prior to the medical breakthrough, $s(x)$ followed de Moivres law with $\omega = 100$ as the limiting age.

   Assuming de Moivres law still applies after the medical breakthrough, calculate the new limiting age.

   (A) 104 (B) 105 (C) 106 (D) 107 (E)108

Solution: For deMoivre's law, $\mathring{e}_{30} = \dfrac{\omega - 30}{2}$.

Prior to medical breakthrough $\omega = 100 \Rightarrow \mathring{e}_{30} = \frac{100-30}{2} = 35$.

After medical breakthrough $\mathring{e}'_{30} = \mathring{e}_{30} + 4 = 39$.

$$\Rightarrow \mathring{e}'_{30} = 39 = \frac{\omega' - 30}{2} \Rightarrow \omega' = 108.$$

Key E

9. For a select-and-ultimate mortality table with a 3-year select period:

   (i)

x	$q_{[x]}$	$q_{[x]+1}$	$q_{[x]+2}$	q_{x+3}	$x+3$
60	0.09	0.11	0.13	0.15	63
61	0.10	0.12	0.14	0.16	64
62	0.11	0.13	0.15	0.17	65
63	0.12	0.14	0.16	0.18	66
64	0.13	0.15	0.17	0.19	67

   (ii) White was a newly selected life on 01/01/2000.

   (iii) White's age on 01/01/2001 is 61.

   (iv) P is the probability on 01/01/2001 that White will be alive on 01/01/2006.

   Calculate P.

   (A) $0 \leq P < 0.43$ (B) $0.43 \leq P < 0.45$ (C) $0.45 \leq P < 0.47$
   (D) $0.47 \leq P < 0.49$ (E) $0.49 \leq P \leq 1.00$

Solution:

$$
\begin{aligned}
{}_5p_{[60]+1} &= \left(1 - q_{[60]+1}\right)\left(1 - q_{[60]+2}\right)(1 - q_{63})(1 - q_{64})(1 - q_{65}) \\
&= (0.89)(0.87)(0.85)(0.84)(0.83) = 0.4589
\end{aligned}
$$

Key: C

10. You are given:

$$
\mu(x) = \begin{cases} 0.04, & 0 < x < 40 \\ 0.05, & x > 40 \end{cases}
$$

Calculate $\overset{\circ}{e}_{25:\overline{25}|}$.

(A) 14.0 (B) 14.4 (C) 14.8 (D) 15.2 (D) 15.6

Solution:

$$
\begin{aligned}
\overset{\circ}{e}_{25:\overline{25}|} &= \int_0^{15} {}_tp_{25}\, dt + {}_{15}p_{25} \int_0^{10} {}_tp_{40}\, dt \\
&= \int_0^{15} e^{-0.04t}\, dt + \left(e^{-\int_0^{15} 0.04\, ds}\right) \int_0^{10} e^{-0.05t}\, dt \\
&= \frac{1}{0.04}\left(1 - e^{-0.60}\right) + e^{-0.60}\left[\frac{1}{0.05}\left(1 - e^{-0.50}\right)\right] \\
&= 11.2797 + 4.3187 = 15.60
\end{aligned}
$$

Key: E

11. T, the future lifetime of (0), has a spliced distribution.

(i) $f_1(t)$ follows the Illustrative Life Table.

(ii) $f_2(t)$ follows DeMoivre's Law with $\omega = 100$.

(iii)

$$
f_T(t) = \begin{cases} k\, f_1(t), & 0 \le t \le 50 \\ 1.2\, f_2(t), & 50 < t \end{cases}
$$

Calculate ${}_{10}p_{40}$.

(A) 0.81 (B) 0.85 (C) 0.88 (D) 0.92 (E) 0.96

SOLUTION: From the Illustrative Life Table:

$$\frac{l_{50}}{l_0} = 0.8951 \quad \frac{l_{40}}{l_0} = 0.9313$$

$$\begin{aligned}
1 = \int_0^\infty f_T(t)\,dt &= \int_0^{50} kf_1(t)\,dt + \int_{50}^\infty 1.2\,f_2(t)\,dt \\
&= kF_1(50) + 1.2\,(F_2(\infty) - F_2(50)) \\
&= k\,(1 - {}_{50}p_0) + 1.2(1 - 0.5) \\
&= k(1 - 0.8951) + 0.6 \\
\Rightarrow k &= \frac{1 - 0.6}{1 - 08951} = 3.813
\end{aligned}$$

For $x \leq 50$,

$$F_T(x) = \int_0^x 3.813 f_1(t)\,dt = 3.813F_1(x)$$

This gives us the following two results:

$$F_T(40) = 3.813\left(1 - \frac{l_{40}}{l_0}\right) = 0.262$$

$$F_T(50) = 3.813\left(1 - \frac{l_{50}}{l_0}\right) = 0.400$$

$${}_{10}p_{40} = \frac{1 - F_T(50)}{1 - F_T(40)} = \frac{1 - 0.400}{1 - 0.262} = 0.813$$

12. For a double decrement table, you are given:

    (i) $\mu_x^{(1)}(t) = 0.2\mu_x^{(\tau)}(t), \quad t > 0$

    (ii) $\mu_x^{(\tau)}(t) = kt^2, \quad t > 0$

    (iii) $q_x^{\prime(1)} = 0.04$

    Calculate ${}_2q_x^{(2)}$.

    (A) 0.45 (B) 0.53 (C) 0.58 (D) 0.64 (E) 0.73

SOLUTION:

We can use the exponential formulation for $p'(x)$

$$0.04 = q_x^{\prime(1)} = 1 - p_x^{\prime(1)} = 1 - e^{-\int_0^1 \mu_x^{(1)}(t)\,dt} = 1 - e^{-\int_0^1 0.2\mu_x^{(\tau)}(t)\,dt} = 1 - e^{-\int_0^1 0.2kt^2\,dt} = 1 - e^{-0.2k/3}$$

$$\Rightarrow e^{-0.2k/3} = 0.96$$

$$\mu_x^{(1)}(t) = 0.2\mu_x^{(\tau)}(t) \quad \Rightarrow \mu_x^{(2)}(t) = 0.8\mu_x^{(\tau)}(t)$$

$$_2q_x^{(2)} = \int_0^2 {}_tp_x^{(\tau)} \cdot \mu_x^{(2)}(t)dt = \int_0^2 {}_tp_x^{(\tau)} \cdot (0.8)\mu_x^{(\tau)}(t)dt = 0.8{}_2q_x^{(\tau)}$$

To get $_2q_x^{(\tau)}$, we use

$$_2q_x^{(\tau)} = 1 - {}_2p_x^{(\tau)} = 1 - e^{-\int_0^1 \mu_x^{(\tau)}(t)\,dt} = 1 - e^{-\int_0^1 kt^2\,dt} = 1 - e^{-8k/3}$$

$$= 1 - \left(e^{-0.2k/3}\right)^{40} = (0.96)^{40} = 0.1954$$

$$\Rightarrow {}_2q_x^{(\tau)} = 0.8046$$

$$\Rightarrow {}_2q_x^{(2)} = (0.8)(0.8046) = 0.644$$

13. You are given:

(i) $\mathring{e}_{30:\overline{40}|} = 27.692$

(ii) $s(x) = 1 - \frac{x}{\omega}$, $x \le \omega$

(iii) $T(x)$ is the future lifetime random variable for (x).

Calculate Var$(T(30))$.

(A) 332

(B) 352

(C) 372

(D) 392

(E) 412

SOLUTION:

$\mathring{e}_{30:\overline{40}|} = \int_0^{40} {}_tp_{30}dt$

$= \int_0^{40} \frac{\omega-30-t}{\omega-30}dt$

$= \left[t - \frac{t^2}{2(\omega-30)}\right]|_0^{40}$
$= 40 - \frac{800}{\omega-30}$
$= 27.692$
$\Rightarrow \omega = 95$

${}_tp_{30} = \frac{65-t}{65}$

Now, realize (after getting $\omega = 95$) that $T(30)$ is uniformly on $(0, 65)$, its variance is just the variance of a continuous uniform random variable:
$Var = \frac{(65-0)^2}{12} = 352.08$
Key: B

14. For a life table with a one-year select period, you are given:

(i)

x	$l_{[x]}$	$d_{[x]}$	l_{x+1}	$\mathring{e}_{[x]}$
80	1000	90	–	8.5
81	920	90	–	–

(ii) Deaths are uniformly distributed over each year of age.

Calculate $\mathring{e}_{[81]}$.

(A) 8.0
(B) 8.1
(C) 8.2
(D) 8.3
(E) 8.4

SOLUTION:

Complete the table:
$l_{81} = l_{[80]} - d_{[80]} = 910$
$l_{82} = l_{[81]} - d_{[81]} = 830$

$p_{[80]} = \frac{910}{1000} = 0.91$
$p_{[81]} = \frac{830}{920} = 0.902$
$p_{81} = \frac{830}{910} = 0.912$
$\dot{e}_{[80]} = \frac{1}{2}q_{[80]} + p_{[80]}\,(1 + \dot{e}_{81})$
where $q_{[80]}$ contributes $\frac{1}{2}$ since UDD
$8.5 = \frac{1}{2}(1 - 0.91) + (0.91)\,(1 + \dot{e}_{81})$

$\dot{e}_{81} = 8.291$
$\dot{e}_{81} = \frac{1}{2}q_{81} + p_{81}\,(1 + \dot{e}_{82})$
$\dot{e}_{82} = 8.043$
$\dot{e}_{[81]} = \frac{1}{2}q_{[81]} + p_{[81]}\,(1 + \dot{e}_{82})$
$= \frac{1}{2}(1 - 0.912) + (0.912)(1 + 8.043)$
$= 8.206$
Key: C

15. You are given:

$$\mu(x) = \begin{cases} 0.05 & 50 \le x < 60 \\ 0.04 & 60 \le x < 70 \end{cases}$$

Calculate ${}_{4|14}q_{50}$.

(A) 0.38

(B) 0.39

(C) 0.41

(D) 0.43

(E) 0.44

SOLUTION:

${}_{4}p_{50} = e^{-(0.05)(4)} = 0.8187$
${}_{10}p_{50} = e^{-(0.05)(10)} = 0.6065$
${}_{8}p_{60} = e^{-(0.04)(8)} = 0.7261$
${}_{18}p_{50} = ({}_{10}p_{50})({}_{8}p_{60}) = 0.6065 \times 0.7261 = 0.4404$

${}_{4|14}q_{50} = {}_{4}p_{50} - {}_{18}p_{50} = 0.8187 - 0.4404 = 0.3783$
Key: A

16. For a population which contains equal numbers of males and females at birth:

(i) For males, $\mu^m(x) = 0.10$, $x \geq 0$
(ii) For females, $\mu^f(x) = 0.08$, $x \geq 0$

Calculate q_{60} for this population.

(A) 0.076
(B) 0.081
(C) 0.086
(D) 0.091
(E) 0.096

SOLUTION:
$s(60) = \frac{e^{-(0.1)(60)} + e^{-(0.08)(60)}}{2} = 0.005354$
$s(61) = \frac{e^{-(0.1)(61)} + e^{-(0.08)(61)}}{2} = 0.00492$
$q_{60} = 1 - \frac{0.00492}{0.005354} = 0.081$
Key: B

Problems from Pre-2000 SOA-CAS exams

We have checked each of these for appropriateness to the current SOA-CAS syllabus. You will notice however, that these questions often have less 'real-life' settings than those on the current exams.

1. You are given: $\mu(x) = \sqrt{\dfrac{1}{80-x}}, \quad 0 \leq x < 80.$

   Calculate the median future lifetime of (20).

   (A) 5.25 (B) 6.08 (C) 8.52 (D) 26.08 (E) 30.00

2. You are given:

   - ${}_tp_x = (0.8)^t, \quad t \geq 0$
   - $l_{x+2} = 6.4$

   Calculate T_{x+1}.

   (A) 4.5 (B) 7.2 (C) 28.7 (D) 35.9 (E) 44.8

Use the following information for the next 4 questions:

You are given:

- $T(x)$ is the random variable for the future lifetime of (x).
- The p.d.f. of T is $f_T(t) = 2e^{-2t}, \quad t \geq 0.$

3. Calculate $\mathring{e}_x$

   (A) 0.5 (B) 2.0 (C) 10.0 (D) 20.0 (E) 40.0

4. Calculate Var$[T]$.

   (A) 0.25 (B) 0.50 (C) 1.00 (D) 2.00 (E) 4.00

5. Calculate $m(x)$, the median future lifetime of (x).

(A) $\dfrac{e^{-4}}{2}$ (B) $\dfrac{e^{-2}}{2}$ (C) $\dfrac{\ln 2}{2}$ (D) $\dfrac{\ln 4}{2}$ (E) 1

6. Calculate m_x, the central-death-rate at age x.

(A) $\dfrac{e^{-2}}{2}$ (B) e^{-2} (C) $2e^{-2}$ (D) 1 (E) 2

7. You are given:

$s(x) = \left(1 - \dfrac{x}{\omega}\right)^{\alpha}, \quad 0 \leq x < \omega, \quad$ where $\alpha > 0$ is a constant.

Calculate $\mu_x \cdot \mathring{e}_x$.

(A) $\dfrac{\alpha}{\alpha+1}$ (B) $\dfrac{\alpha\omega}{\alpha+1}$ (C) $\dfrac{\alpha^2}{\alpha+1}$ (D) $\dfrac{\alpha^2}{\omega - x}$ (E) $\dfrac{\alpha(\omega - x)}{(\alpha+1)\omega}$

8. You are given:

- $q_{60} = 0.3$
- $q_{61} = 0.4$
- f is the probability that (60) will die between ages 60.5 and 61.5 under the uniform distribution of deaths assumption.
- g is the probability that (60) will die between ages 60.5 and 61.5 under the Balducci assumption.

Calculate $10{,}000(g - f)$.

(A) 0 (B) 85 (C) 94 (D) 178 (E) 213

Solutions to Chapter 3

1. **Key: A** We need to find t such that ${}_tp_{20} = 0.5$

$$\Rightarrow 0.5 = e^{-\int_0^t \mu(20+s)\,ds} = e^{-\int_0^t (60-s)^{-0.5}\,ds}$$

$$= e^{2\left[(60-s)^{0.5}\right]_0^t} = e^{2\left[\sqrt{60-t}-\sqrt{60}\right]}$$

$$\ln 0.5 = 2\left[\sqrt{60-t}-\sqrt{60}\right]$$

$$\Rightarrow \sqrt{60-t} = 7.40 \quad \Rightarrow t = 5.25$$

2. **Key: D**

$$T_{x+1} = \int_0^\infty l_{x+1+t}\,dt$$

$$l_{x+1} = \frac{6.4}{0.8} = 8 \quad \Rightarrow l_{x+1+t} = 8(0.8)^t$$

$$T_{x+1} = 8\int_0^\infty (0.8)^t\,dt = \frac{8}{\ln 0.8}(0.8)^t|_0^\infty$$

$$= \frac{-8}{\ln 0.8} = 35.9$$

3. **Key: A** This is constant force of mortality (CFM) with $\mu = 2$.

$$\mathrm{E}[T] = \frac{1}{\mu} = 0.5 = \mathring{e}_x$$

4. **Key: A** (CFM) $\mathrm{Var}[T] = \frac{1}{\mu^2} = 0.25$

5. **Key: C**

$$0.5 = {}_tp_x = e^{-\mu\cdot t} = e^{-2t}$$

$$\Rightarrow t = \frac{\ln 2}{2}$$

6. **Key: E**

$$m_x = \frac{\ell_x - \ell_{x+1}}{L_x} = \frac{\ell_x(1-e^{-\mu})}{\int_0^1 \ell_x \cdot {}_tp_x\,dt}$$
$$= \frac{\ell_x(1-e^{-2})}{\ell_x \int_0^1 e^{-\mu t}\,dt} == \frac{1-e^{-2}}{\frac{1}{2}(1-e^{-2})} = 2$$

7. **Key: A** This is a Modified DeMoivre problem with constant equal to α. So

$$\mu(x) = \frac{\alpha}{\omega - x}, \qquad \mathring{e}_x = \frac{\omega - x}{\alpha + 1}$$

Multiplying these two quantities gives

$$\frac{\alpha}{\alpha+1}$$

8. **Key: B** In both cases the probability is given by

$${}_{0.5}p_{60} \cdot {}_{0.5}q_{60.5} + {}_{1}p_{60} \cdot {}_{0.5}q_{61}$$

UDD: $[1-(0.5)(0.3)]\,\frac{(0.5)(0.3)}{1-(0.5)(0.3)} + (0.7)\cdot(0.5)(0.4) = 0.2900 = f$

Balducci: $\frac{0.7}{1-(0.5)(0.3)}\,\frac{(0.5)(0.3)}{1} + (0.7)\cdot\frac{(0.5)(0.4)}{1-(0.5)(0.4)} = 0.2985 = g$

$$10{,}000(g-f) = 85$$