Actuarial Mathematics: Chapter 3 – Suggested Problems and Solutions.

Suggested Problems:

1(do first row last), 5, 6, 7, 9, 12, 18abc, 20, 28, 30, 36, 39

Suggested Solutions:

1. Except for the first row, these are pretty straightforward using the table in this manual at the beginning of Section 3.2.4. The first row is worked below.

$$s(x) = e^{-\int_0^x \tan t dt}$$

The integral can be written as

$$\int_0^x \frac{\sin t}{\cos t} dt = \int_0^x \frac{-du}{u} = -\ln |u|_0^x$$

(where $u = \cos t$ and $du = -\sin t$)

$$= -\ln(\cos x) + \ln(\cos 0) = -\ln(\cos x).$$

We can drop the absolute values here because we are told everything is between 0 and $\frac{\pi}{2}$. Now

$$s(x) = e^{\ln(\cos x)} = \cos x,$$

$$\Rightarrow F(x) = 1 - \cos x, \quad f(x) = -s'(x) = \sin x.$$

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5. Note that $s(x) = \frac{100-x}{100}$, DeMoivre's Law.

(A) $\mu(x) = -s'(x)/s(x) = \frac{1}{100-x}$

(B) $F(x) = 1 - s(x) = \frac{x}{100}$

(C) $f(x) = -s'(x) = \frac{1}{100}$

(D) $s(10) - s(40) = \frac{90}{100} - \frac{60}{100} = 0.3.$

6.

(A) ${}_tp_{40} = \frac{s(40+t)}{s(40)} = \frac{60-t}{60}$, for DeMoivre's Law, you should memorize that ${}_tp_x = \frac{\omega-x-t}{\omega-x}$.

(B)

$$\frac{-\frac{d}{dt}({}_tp_{40})}{{}_tp_{40}} = \frac{1}{60-t}.$$

For DeMoivre, $\mu_x(t) = \frac{1}{\omega-x-t}$.

(C) $= -\frac{d}{dt}({}_tp_{40}) = \frac{1}{60}$, DeMoivre's law assumes a uniform distribution – death is equally likely at any time from x to ω.

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7. See important note at the end of this problem.

(A)

$$\frac{s(36)}{s(19)} = \left(\frac{100-36}{100} \cdot \frac{100}{100-19}\right)^{0.5} = \left(\frac{100-36}{100-19}\right)^{0.5} = \frac{8}{9}$$

(B) $1 - \frac{s(51)}{s(36)} = 1 - \left(\frac{100-51}{100-36}\right)^{0.5} = \frac{1}{8}$

(C) ${}_{15|13}q_{36} = {}_{15}p_{36}\ {}_{13}q_{51} = \frac{s(51)}{s(36)}\left(1 - \frac{s(64)}{s(51)}\right) = \frac{7}{8} \cdot \frac{1}{7} = \frac{1}{8}$.

(D) Note that

$$s(x) = \left(\frac{100-x}{100}\right)^{\frac{1}{2}} \implies s'(x) = -\frac{1}{200}\left(\frac{100-x}{100}\right)^{-\frac{1}{2}}.$$

$$\mu(x) = \frac{-s'(x)}{s(x)} = \frac{1}{2}\left(\frac{1}{100-x}\right),$$

$$\Rightarrow \mu(36) = \frac{1}{128}.$$

(E) According to the survival function, (36) can survive at most 64 more years, so

$$\mathring{e}_{36} = \int_0^{64} {}_tp_{36}\, dt = \int_0^{64} \frac{s(36+t)}{s(36)}\, dt$$

$$s(36+t) = \left(\frac{64-t}{100}\right)^{\frac{1}{2}}, \quad s(36) = 0.8$$

$$\Rightarrow \mathring{e}_{36} = \frac{1}{8}\int_0^{64} (64-t)^{\frac{1}{2}}\, dt = \frac{1}{8}\left[\frac{-2}{3}(64-t)^{\frac{3}{2}}\right]_0^{64} = \frac{1}{8}\,\frac{2}{3}\,8^3 = \frac{128}{3}$$

Important Note: This is a Modified DeMoivre's Law with

$$s(x) = \left(\frac{\omega - x}{\omega}\right)^{\alpha},$$

instead of the DeMoivre we are used to ($\alpha = 1$). For Modified DeMoivre,

$$\mu(x) = \frac{\alpha}{\omega - x}, \quad \overset{\circ}{e}_x = \frac{\omega - x}{\alpha + 1}.$$

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9. This is constant force of mortality. ${}_tp_x = e^{-0.001t}$ for any x.

$${}_{2|2}q_{20} = {}_2p_{20}\,{}_2q_{22} = e^{-0.002}(1 - e^{-0.002}) = 0.001994$$

12.

(A) ${}_5q_0 = 1 - \frac{\ell_5}{\ell_0} = 1 - \frac{98495}{100000} = 0.01505.$

${}_5q_5 = 1 - \frac{\ell_{10}}{\ell_5} = 1 - \frac{98347}{98495} = 0.001503.$ Fewer die in the second 5 years of life than in the first 5 years of life.

(B) ${}_{55|5}q_{25} = \dfrac{\text{number dying between 80 and 85}}{\ell_{25}} = \frac{\ell_{80} - \ell_{85}}{\ell_{25}} = \frac{43180 - 27960}{97110} = 0.1567$

18.

(A) $s(x) = \int_x^{\infty} f(x) = e^{-cx}$. This is constant force with force of mortality $\mu = c$, so $\overset{\circ}{e}_x = \frac{1}{\mu} = \frac{1}{c}$.

(B)

$$\text{Var}[T] = 2\int_0^{\infty} t \cdot {}_tp_x\, dt - \left(\overset{\circ}{e}_x\right)^2 = 2\int_0^{\infty} te^{-ct} dt - \left(\overset{\circ}{e}_x\right)^2$$

For the integral, we can use integration by parts with

$$\begin{array}{ll} u = t & du = dt \\ v = -\frac{1}{c}e^{-ct} & dv = e^{-ct}dv \end{array}$$

Which gives that the integral equals

$$2\left[\frac{-t}{c}e^{-ct}\right]_0^{\infty} + \frac{2}{c}\int_0^{\infty} e^{-ct}\, dt = 0 - \frac{2}{c^2}[e^{-ct}]_0^{\infty} = \frac{2}{c^2}$$

$$\Rightarrow \text{Var}[T] = \frac{2}{c^2} - \frac{1}{c^2} = \frac{1}{c^2}.$$

For CFM, $\text{Var}[T] = \frac{1}{\mu^2}$.

(C) Find t such that ${}_tp_x = 0.5$

$$\Rightarrow e^{-ct} = 0.5 \quad \Rightarrow \qquad t = \frac{\ln 2}{c}$$

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20. $f_x(t) = F'_{T(x)}(t) = \frac{1}{100-x}$, this is DeMoivre's Law with $\omega = 100$

(A) $\mathring{e}_x = \frac{\omega - x}{2} = \frac{100-x}{2}$ since this is DeMoivre's law.
You could also work out

$$\mathring{e}_x = \int_0^{100-x} {}_tp_x dt = \int_0^{100-x} \frac{100-x-t}{100-x} dt$$

(B)

$$\mathrm{Var}[T] = 2\int_0^{\infty} t \cdot {}_tp_x \, dt - \mathring{e}_x^2$$

The integral is

$$2\int_0^{\infty} t \cdot \frac{100-x-t}{100-x}\, dt = \frac{2}{100-x}\int_0^{100-x} (100-x)t - t^2 \, dt$$

A polynomial!

$$= \frac{2}{100-x}\left[\frac{100-x}{2}t^2 - \frac{t^3}{3}\right]_0^{100-x} = \frac{2}{100-x}\left[\frac{(100-x)^3}{2} - \frac{(100-x)^3}{3}\right] = \frac{1}{3}(100-x)^2.$$

$$\Rightarrow \mathrm{Var}[T] = \frac{(100-x)^2}{3} - \frac{(100-x)^2}{4} = \frac{(100-x)^2}{12}$$

To avoid doing this calculation in the future, remember that, for DeMoivre's Law,

$$\mathrm{Var}[T] = \frac{(\omega - x)^2}{12}$$

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28. For all three assumptions, we will use the fact that

$$q_{65} = 1 - p_{65} = 1 - \frac{\ell_{66}}{\ell_{65}} = 1 - \frac{75520}{77107} = 0.02058.$$

(A) Uniform assumption: ${}_{0.5}q_x = (0.5)q_x = 0.01029$, which implies that ${}_{0.5}p_{65} = 0.98971$.

(B) Constant Force:

$${}_{0.5}p_{65} = p_{65}^{(0.5)} = (0.97942)^{0.5} = 0.98966$$

(C) Hyperbolic assumption (Balducci)

$$ {}_{0.5}p_{65} = \frac{p_{65}}{1-(1-0.5)q_{65}} = \frac{0.97942}{1-0.01029} = 0.98960. $$

Balducci is a pain!

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30. For both of these, the quantity we are seeking is

$$ {}_{0.5}p_{70} \cdot {}_{1}q_{70.5} = {}_{0.5}p_{70}({}_{0.5}q_{70.5} + {}_{0.5}p_{70.5} \cdot {}_{0.5}q_{71}) $$

(A)

$$ {}_{0.5}p_{70} = 1 - {}_{0.5}q_{70} = 1-(0.5)q_{70} = 0.98 $$

$$ {}_{0.5}q_{70.5} = \frac{(1-0.5)q_{70}}{1-(0.5)q_{70}} = \frac{0.02}{0.98} = 0.0204, \qquad {}_{0.5}p_{70.5} = 0.9796 $$

$$ {}_{0.5}q_{71} = (0.5)q_{71} = 0.025 $$

$$ \Rightarrow {}_{0.5}p_{70}({}_{0.5}q_{70.5} + {}_{0.5}p_{70.5} \cdot {}_{0.5}q_{71}) = 0.98[0.0204 + (0.9796)(0.025)] = 0.04399 $$

(B)

$$ {}_{0.5}p_{70} = \frac{p_{70}}{1-(1-0.5)q_{70}} = \frac{0.96}{0.98} = 0.9796 $$

$$ {}_{0.5}q_{70.5} = (1-0.5)q_{70} = 0.02, \qquad {}_{0.5}p_{70.5} = 0.98 $$

$$ {}_{0.5}q_{71} = \frac{(0.5)q_{71}}{1-(1-0.5)q_{71}} = \frac{0.025}{0.975} = 0.02564 $$

$$ \Rightarrow {}_{0.5}p_{70}({}_{0.5}q_{70.5} + {}_{0.5}p_{70.5} \cdot {}_{0.5}q_{71}) = 0.9796[0.02 + (0.98)(0.02564)] = 0.04421 $$

36.

(A)

$$ {}_{2}q_{[32]+1} = 1 - \frac{\ell_{[32]+3}}{\ell_{[32]+1}} = 1 - \frac{\ell_{35}}{\ell_{[32]+1}} = 1 - \frac{9887.6}{9896.3} = 0.000879. $$

The book's answer is different and seems way too big. Same for part B.

(B)

$$ {}_{2}p_{[31]+1} = \frac{\ell_{34}}{\ell_{[31]+1}} = \frac{9892.5}{9900.6} = 0.999182 $$

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39. This question is so 'exam-like' that you might want to work it twice. Let p_x refer to standard mortality and p_x^* refer to standard mortality plus the extra risk.

$$0.994 = p_{50} = e^{-\int_0^1 \mu_x(t)\,dt}$$

We need a function for the extra mortality to add to calculate p_x^*. The linear function that equals 0.03 at $t = 0$ and 0.0 at $t = 1$ is $\mu_x^*(t) = 0.03(1 - t)$, so

$$p_{50}^* = e^{-\int_0^1 [\mu_x(t)+(0.03)(1-t)]\,dt} = \left(e^{-\int_0^1 \mu_x(t)\,dt}\right)\left(e^{-\int_0^1 (0.03)(1-t)\,dt}\right) = (0.994)\left(e^{-\int_0^1 (0.03)(1-t)\,dt}\right)$$

The integral in the rightmost expression is

$$0.03\left[t - \frac{t^2}{2}\right]_0^1 = 0.015.$$

$$\Rightarrow p_{50}^* = (0.994)\left(e^{-0.015}\right) = 0.9792$$