Actuarial Mathematics: Chapter 3 – Suggested Problems and Solutions.

Suggested Problems:

1(do first row last), 5, 6, 7, 9, 12, 18abc, 20, 30, 36, 39

Suggested Solutions:

1. Except for the first row, these are pretty straightforward using the table in this manual at the beginning of Section 3.2.4. The first row is worked below.

$$s(x) = e^{-\int_0^x \tan t dt}$$

The integral can be written as

$$\int_0^x \frac{\sin t}{\cos t} dt = \int_0^x \frac{-du}{u} = -\ln |u|_0^x$$

(where $u = \cos t$ and $du = -\sin t$)

$$= -\ln(\cos x) + \ln(\cos 0) = -\ln(\cos x).$$

We can drop the absolute values here because we are told everything is between 0 and $\frac{\pi}{2}$. Now

$$s(x) = e^{\ln(\cos x)} = \cos x,$$

$$\Rightarrow F(x) = 1 - \cos x, \quad f(x) = -s'(x) = \sin x.$$

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5. Note that $s(x) = \frac{100-x}{100}$, DeMoivre's Law.

(A) $\mu(x) = -s'(x)/s(x) = \frac{1}{100-x}$

(B) $F(x) = 1 - s(x) = \frac{x}{100}$

(C) $f(x) = -s'(x) = \frac{1}{100}$

(D) $s(10) - s(40) = \frac{90}{100} - \frac{60}{100} = 0.3.$

6.

(A) ${}_tp_{40} = \frac{s(40+t)}{s(40)} = \frac{60-t}{60}$, for DeMoivre's Law, you should memorize that ${}_tp_x = \frac{\omega-x-t}{\omega-x}$.

(B)

$$\frac{-\frac{d}{dt}({}_tp_{40})}{{}_tp_{40}} = \frac{1}{60-t}.$$

For DeMoivre, $\mu_x(t) = \frac{1}{\omega-x-t}$.

(C) $= -\frac{d}{dt}({}_tp_{40}) = \frac{1}{60}$, DeMoivre's law assumes a uniform distribution – death is equally likely at any time from x to ω.

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7. See important note at the end of this problem.

(A)

$$\frac{s(36)}{s(19)} = \left(\frac{100-36}{100} \cdot \frac{100}{100-19}\right)^{0.5} = \left(\frac{100-36}{100-19}\right)^{0.5} = \frac{8}{9}$$

(B) $1 - \frac{s(51)}{s(36)} = 1 - \left(\frac{100-51}{100-36}\right)^{0.5} = \frac{1}{8}$

(C) ${}_{15|13}q_{36} = {}_{15}p_{36}\;{}_{13}q_{51} = \frac{s(51)}{s(36)}\left(1 - \frac{s(64)}{s(51)}\right) = \frac{7}{8} \cdot \frac{1}{7} = \frac{1}{8}$.

(D) Note that

$$s(x) = \left(\frac{100-x}{100}\right)^{\frac{1}{2}} \implies s'(x) = -\frac{1}{200}\left(\frac{100-x}{100}\right)^{-\frac{1}{2}}.$$

$$\mu(x) = \frac{-s'(x)}{s(x)} = \frac{1}{2}\left(\frac{1}{100-x}\right),$$

$$\Rightarrow \mu(36) = \frac{1}{128}.$$

(E) According to the survival function, (36) can survive at most 64 more years, so

$$\overset{\circ}{e}_{36} = \int_0^{64} {}_tp_{36}\,dt = \int_0^{64} \frac{s(36+t)}{s(36)}\,dt$$

$$s(36+t) = \left(\frac{64-t}{100}\right)^{\frac{1}{2}}, \quad s(36) = 0.8$$

$$\Rightarrow \overset{\circ}{e}_{36} = \frac{1}{8}\int_0^{64}(64-t)^{\frac{1}{2}}\,dt = \frac{1}{8}\left[\frac{-2}{3}(64-t)^{\frac{3}{2}}\right]_0^{64} = \frac{1}{8}\,\frac{2}{3}\,8^3 = \frac{128}{3}$$

Important Note: This is a Modified DeMoivre's Law with

$$s(x) = \left(\frac{\omega - x}{\omega}\right)^{\alpha},$$

instead of the DeMoivre we are used to ($\alpha = 1$). For Modified DeMoivre,

$$\mu(x) = \frac{\alpha}{\omega - x}, \quad \mathring{e}_x = \frac{\omega - x}{\alpha + 1}.$$

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9. This is constant force of mortality. ${}_tp_x = e^{-0.001t}$ for any x.

$${}_{2|2}q_{20} = {}_2p_{20}\,{}_2q_{22} = e^{-0.002}(1 - e^{-0.002}) = 0.001994$$

12.

(A) ${}_5q_0 = 1 - \frac{\ell_5}{\ell_0} = 1 - \frac{98495}{100000} = 0.01505.$

${}_5q_5 = 1 - \frac{\ell_{10}}{\ell_5} = 1 - \frac{98347}{98495} = 0.001503$. Fewer die in the second 5 years of life than in the first 5 years of life.

(B) ${}_{55|5}q_{25} = \dfrac{\text{number dying between 80 and 85}}{\ell_{25}} = \frac{\ell_{80} - \ell_{85}}{\ell_{25}} = \frac{43180 - 27960}{97110} = 0.1567$

18.

(A) $s(x) = \int_x^\infty f(x) = e^{-cx}$. This is constant force with force of mortality $\mu = c$, so $\mathring{e}_x = \frac{1}{\mu} = \frac{1}{c}$.

(B)

$$\text{Var}[T] = 2\int_0^\infty t \cdot {}_tp_x\, dt - \left(\mathring{e}_x\right)^2 = 2\int_0^\infty te^{-ct} dt - \left(\mathring{e}_x\right)^2$$

For the integral, we can use integration by parts with

$$u = t \qquad du = dt$$
$$v = -\tfrac{1}{c}e^{-ct} \qquad dv = e^{-ct}dv$$

Which gives that the integral equals

$$2\left[\frac{-t}{c}e^{-ct}\right]_0^\infty + \frac{2}{c}\int_0^\infty e^{-ct}\, dt = 0 - \frac{2}{c^2}[e^{-ct}]_0^\infty = \frac{2}{c^2}$$

$$\Rightarrow \text{Var}[T] = \frac{2}{c^2} - \frac{1}{c^2} = \frac{1}{c^2}.$$

For CFM, $\text{Var}[T] = \frac{1}{\mu^2}$.

(C) Find t such that ${}_tp_x = 0.5$

$$\Rightarrow e^{-ct} = 0.5 \quad \Rightarrow \qquad t = \frac{\ln 2}{c}$$

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20. $f_x(t) = F'_{T(x)}(t) = \frac{1}{100-x}$, this is DeMoivre's Law with $\omega = 100$

(A) $\overset{\circ}{e}_x = \frac{\omega - x}{2} = \frac{100-x}{2}$ since this is DeMoivre's law.
You could also work out

$$\overset{\circ}{e}_x = \int_0^{100-x} {}_tp_x dt = \int_0^{100-x} \frac{100-x-t}{100-x} dt$$

(B)

$$\text{Var}[T] = 2\int_0^{\infty} t \cdot {}_tp_x\, dt - \overset{\circ}{e}{}_x^2$$

The integral is

$$2\int_0^{\infty} t \cdot \frac{100-x-t}{100-x}\, dt = \frac{2}{100-x}\int_0^{100-x} (100-x)t - t^2\, dt$$

A polynomial!

$$= \frac{2}{100-x}\left[\frac{100-x}{2}t^2 - \frac{t^3}{3}\right]_0^{100-x} = \frac{2}{100-x}\left[\frac{(100-x)^3}{2} - \frac{(100-x)^3}{3}\right] = \frac{1}{3}(100-x)^2.$$

$$\Rightarrow \text{Var}[T] = \frac{(100-x)^2}{3} - \frac{(100-x)^2}{4} = \frac{(100-x)^2}{12}$$

To avoid doing this calculation in the future, remember that, for DeMoivre's Law,

$$\text{Var}[T] = \frac{(\omega - x)^2}{12}$$

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30. For both of these, the quantity we are seeking is

$${}_{0.5}p_{70} \cdot {}_1q_{70.5} = {}_{0.5}p_{70}({}_{0.5}q_{70.5} + {}_{0.5}p_{70.5} \cdot {}_{0.5}q_{71})$$

(A)

$${}_{0.5}p_{70} = 1 - {}_{0.5}q_{70} = 1 - (0.5)q_{70} = 0.98$$

$${}_{0.5}q_{70.5} = \frac{(1-0.5)q_{70}}{1-(0.5)q_{70}} = \frac{0.02}{0.98} = 0.0204, \qquad {}_{0.5}p_{70.5} = 0.9796$$

$${}_{0.5}q_{71} = (0.5)q_{71} = 0.025$$

$$\Rightarrow {}_{0.5}p_{70}({}_{0.5}q_{70.5} + {}_{0.5}p_{70.5} \cdot {}_{0.5}q_{71}) = 0.98[0.0204 + (0.9796)(0.025)] = 0.04399$$

(B)

$$
{}_{0.5}p_{70} = \frac{p_{70}}{1-(1-0.5)q_{70}} = \frac{0.96}{0.98} = 0.9796
$$

$$
{}_{0.5}q_{70.5} = (1-0.5)q_{70} = 0.02, \qquad {}_{0.5}p_{70.5} = 0.98
$$

$$
{}_{0.5}q_{71} = \frac{(0.5)q_{71}}{1-(1-0.5)q_{71}} = \frac{0.025}{0.975} = 0.02564
$$

$$
\Rightarrow {}_{0.5}p_{70}({}_{0.5}q_{70.5} + {}_{0.5}p_{70.5} \cdot {}_{0.5}q_{71}) = 0.9796[0.02 + (0.98)(0.02564)] = 0.04421
$$

36.

(A)

$$
{}_{2}q_{[32]+1} = 1 - \frac{\ell_{[32]+3}}{\ell_{[32]+1}} = 1 - \frac{\ell_{35}}{\ell_{[32]+1}} = 1 - \frac{9887.6}{9896.3} = 0.000879.
$$

The book's answer is different and seems way too big. Same for part B.

(B)

$$
{}_{2}p_{[31]+1} = \frac{\ell_{34}}{\ell_{[31]+1}} = \frac{9892.5}{9900.6} = 0.999182
$$

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39. This question is so 'exam-like' that you might want to work it twice. Let p_x refer to standard mortality and p_x^* refer to standard mortality plus the extra risk.

$$
0.994 = p_{50} = e^{-\int_0^1 \mu_x(t)\,dt}
$$

We need a function for the extra mortality to add to calculate p_x^*. The linear function that equals 0.03 at $t = 0$ and 0.0 at $t = 1$ is $\mu_x^*(t) = 0.03(1-t)$, so

$$
p_{50}^* = e^{-\int_0^1 [\mu_x(t)+(0.03)(1-t)]\,dt} = \left(e^{-\int_0^1 \mu_x(t)\,dt}\right)\left(e^{-\int_0^1 (0.03)(1-t)\,dt}\right) = (0.994)\left(e^{-\int_0^1 (0.03)(1-t)\,dt}\right)
$$

The integral in the rightmost expression is

$$
0.03\left[t - \frac{t^2}{2}\right]_0^1 = 0.015.
$$

$$
\Rightarrow p_{50}^* = (0.994)\left(e^{-0.015}\right) = 0.9792
$$

Actuarial Mathematics: Chapter 4 – Suggested Problems and Solutions.

Suggested Problems:

6, 7, 10, 11ac, 14a, 16, 23, 26a

Suggested Solutions:

6. This is DeMoivre's Law:

(A)

$$ {}_tp_x = \frac{\omega - x - t}{\omega - x}, \qquad \mu_x(t) = \frac{1}{\omega - x - t} $$

$$ \overline{A}^{\,1}_{40:\overline{25}|} = \int_0^{25} v^t \, {}_tp_x \, \mu_x(t)\, dt = \int_0^{25} e^{-0.05t} \frac{\omega - x - t}{\omega - x} \frac{1}{\omega - x - t}\, dt $$

$$ = \frac{1}{\omega - x}\int_0^{25} e^{-0.05t} dt = \frac{1}{(0.05)(\omega - x)}\left[1 - e^{-1.25}\right] = \frac{1}{3}(0.7135) = 0.238 $$

(B)

$$ \int_0^{25} b_t \, v^t \, {}_tp_x \, \mu_x(t)\, dt = \int_0^{25} e^{0.05t}\, e^{-0.05t} \frac{\omega - x - t}{\omega - x} \frac{1}{\omega - x - t}\, dt = \int_0^{25} \frac{1}{\omega - x}\, dt $$

$$ = \frac{25}{60} = \frac{5}{12}. $$

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7. More DeMoivre with $\delta = \ln(1+i) = 0.0953$:

(A)

$$ \overline{A}^{\,1}_{30:\overline{10}|} = \int_0^{10} v^t \, {}_tp_x \, \mu_x(t)\, dt = \int_0^{10} e^{-0.0953t} \frac{\omega - x - t}{\omega - x} \frac{1}{\omega - x - t}\, dt $$

$$ = \frac{1}{\omega - x}\int_0^{10} e^{-0.0953t} dt = \frac{1}{(0.0953)(\omega - x)}\left[1 - e^{-0.953}\right] = \frac{10.493}{70}(0.6144) = 0.0921 $$

(B)

$$ \text{Var}[Z] = {}^2\overline{A}^{\,1}_{30:\overline{10}|} - \left(\overline{A}^{\,1}_{30:\overline{10}|}\right)^2 $$

We can use the rule of moments since the benefit is always 0 or 1 (1 if the insured dies within 10 years, and 0 otherwise.) For the first term, we double δ and do the same calculation:

$$ {}^2\overline{A}^{\,1}_{30:\overline{10}|} = \frac{1}{(0.1906)(\omega - x)}\left[1 - e^{-1.906}\right] = 0.0638 $$

$$^2\overline{A}_{\stackrel{1}{30}:\overline{10}|} - \left(\overline{A}_{\stackrel{1}{30}:\overline{10}|}\right)^2 = 0.0638 - 0.00848 = 0.0553$$

10.

(A)

$$(\overline{IA})_x = \int_0^\infty t\, e^{-\delta t}\, {}_tp_x\, \mu\, dt = \int_0^\infty t\, e^{-(\delta+\mu)t}\, \mu\, dt$$

Using integration by parts this integral is equal to

$$\left[\frac{-t\mu}{\mu+\delta} e^{-(\delta+\mu)t}\right]_0^\infty + \frac{\mu}{\mu+\delta}\int_0^\infty e^{-(\mu+\delta)t}\, dt = 0 + \frac{\mu}{(\mu+\delta)^2}$$

(B) We can't use the rule of moments for this one because the benefit takes on values other than 0 or 1.

$$\text{Var}[Z] = \text{E}\left[Z^2\right] - (\text{E}[Z])^2$$

We have E[Z] from part (A).

$$\text{E}\left[Z^2\right] = \int_0^\infty t^2\, e^{-2\delta t}\, {}_tp_x\, \mu\, dt = \int_0^\infty t^2\, e^{-(\mu+2\delta)t}\, \mu\, dt.$$

Now, doing 2 sets of integration by parts gives the answer in the book.

11a). Start by finding the CDF of Z. To do that, we start by writing down its definition:

$$F(z) = \Pr\left[Z < z\right]$$

Since we are used to dealing with T, put everything in terms of T.

$$\Pr\left[Z < z\right] = \Pr\left[v^T < z\right] = \Pr\left[e^{-\delta T} < z\right] = \Pr\left[-\delta T < \ln z\right]$$

$$= \Pr\left[T > \frac{-\ln z}{\delta}\right] = e^{-\mu\left(\frac{-\ln z}{\delta}\right)} = \left(e^{\ln z}\right)^{\frac{\mu}{\delta}} = z^{\frac{\mu}{\delta}},$$

where we used the fact that $\Pr[T > x] = e^{-\mu x}$ since we are dealing with constant force of mortality. So the pdf of Z is

$$f_Z(z) = F'(z) = \frac{\mu}{\delta}\left(z^{\frac{\mu}{\delta}-1}\right) = 0.2z^{-0.8}$$

c). Constant force of mortality:

$$\text{E}[Z] = \frac{\mu}{\mu+\delta} = 0.1667.$$

$$\text{Var}[Z] = {}^2A_x - (A_x)^2 = \frac{\mu}{\mu + 2\delta} - \frac{\mu^2}{(\mu+\delta)^2} = 0.0909 - 0.0278 = 0.0631.$$

I don't know where the answers in the book came from.

14a). DeMoivre

$$A_{40:\overline{25|}} = A^{1}_{40:\overline{25|}} + A_{40:\overset{1}{\overline{25|}}}$$

$$A_{40:\overset{1}{\overline{25|}}} = v^{25}\,{}_{25}p_{40} = \left(\frac{1}{1.05}\right)^{25}\frac{\omega - 40 - 25}{60} = 0.172$$

$$A^{1}_{40:\overline{25|}} = \sum_{k=0}^{\infty} {}_kp_x\, q_{x+k}\, v^{k+1} = \sum_{k=0}^{\infty}\left(\frac{\omega - x - k}{\omega - x}\right)\frac{1}{\omega - x - k}\left(\frac{1}{1.05}\right)^{k+1}$$

$$= \frac{1}{60}\sum_{k=1}^{25}(0.9524)^k = \frac{1}{60}(0.9524)\frac{1-(0.9524)^{25}}{1-0.9524} = \frac{14.1}{60} = 0.235$$

$$\Rightarrow A_{40:\overline{25|}} = 0.172 + 0.235 = 0.407.$$

16.

$$0.55 = A_{x:\overline{20|}} = A^{1}_{x:\overline{20|}} + A_{x:\overset{1}{\overline{20|}}}$$

$$0.25 = A_x = A^{1}_{x:\overline{20|}} + A_{x:\overset{1}{\overline{20|}}}A_{x+20}$$

$$0.55 = A^{1}_{x:\overline{20|}} + A_{x:\overset{1}{\overline{20|}}}$$

$$0.25 = A^{1}_{x:\overline{20|}} + 0.4A_{x:\overset{1}{\overline{20|}}}$$

Subtracting these two equations gives

$$0.3 = 0.6A_{x:\overline{20|}} \qquad \Rightarrow A_{x:\overline{20|}} = 0.50$$

$$\Rightarrow A^{1}_{x:\overline{20|}} = 0.05$$

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23. This insurance is equivalent to a term policy to age 65, plus a whole life policy –

$$A^{1}_{x:\overline{65-x|}} + A_x.$$

26a. The hard part about this one is getting the information they are giving us down on paper. We are told the following:

$$700 = 1000A_{x:\overset{1}{\overline{n|}}} + 700A^{1}_{x:\overline{n|}}$$

$$650 = 1000A_{x:\overset{1}{\overline{n|}}}$$

$$\Rightarrow A_{\overset{1}{x}:\overline{n}|} = \frac{1}{14}, \qquad A_{x:\overset{1}{\overline{n}|}} = 0.65$$

We are supposed to find the actuarial present value A of an insurance such that

$$A = 1000A_{x:\overset{1}{\overline{n}|}} + kA \cdot A_{\overset{1}{x}:\overline{n}|}.$$

$$\Rightarrow A = 650 + kA\left(\frac{1}{14}\right)$$

$$\Rightarrow A = \frac{9100}{14-k}.$$

Actuarial Mathematics: Chapter 5 – Suggested Problems and Solutions.

Suggested Problems:

1, 6a, 33, 45, 51, 53(age 13 only), 56(age 13 only),

Additional Problem:

(A) Find an expression for $\ddot{a}_x - a_x$.

(B) Find an expression for $\ddot{a}_{x:\overline{n}|} - a_{x:\overline{n}|}$.

(C) $i = 0.08$, $\ddot{a}_{66:\overline{10}|} = 6$, $p_{65} = 0.95$. **Find $a_{65:\overline{10}|}$.**

(D) Use the illustrative life table to find $a_{30:\overline{20}|}$.

Suggested Solutions:

1.

(A)

$$\overline{A}_{20} = \frac{i}{\delta} A_{20} = \frac{0.06}{\ln 1.06} 0.06528 = 0.06722$$

$$\overline{a}_{20} = \frac{1 - \overline{A}_{20}}{\delta} = \frac{0.9328}{0.05827} = 16.01$$

(B)

$$\text{Var}\left[\overline{a}_{\overline{T}|}\right] = \frac{{}^2\overline{A}_{20} - \left(\overline{A}_{20}\right)^2}{\delta^2}$$

To find ${}^2\overline{A}_{20}$, we double the force of interest everwhere. To do this correctly, we have to note that $1 + i = e^{\delta} \Rightarrow (1 + 2i + i^2) = e^{2\delta}$. So doubling the force of interest is equivalent to replacing i with $(2i + i^2)$.

$$\Rightarrow {}^2\overline{A}_{20} = \frac{2i + i^2}{2\delta} \cdot {}^2A_{20} = \frac{0.1236}{0.1165}(0.014303) = 0.01517$$

$$\Rightarrow \text{Var}\left[\overline{a}_{\overline{T}|}\right] = \frac{(0.01517) - (0.06722)^2}{(0.05827)^2} = 3.137$$

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6. (A) Start by writing down the definition of the cumulative distribution function, and then get everything in terms of T, which we alread know how to work with.

$$F_Y(y) = \Pr[Y < y] = \Pr\left[\frac{1 - v^T}{\delta} < y\right] = \Pr\left[-v^T < \delta y - 1\right]$$

But v^T is just $e^{-\delta t}$, so the probability equals

$$\Pr\left[e^{-\delta T} > 1 - \delta y\right] = \Pr\left[-\delta T > \ln(1-\delta y)\right] = \Pr\left[T < \frac{\ln(1-\delta y)}{-\delta}\right]$$

But $\Pr[T < a] = 1 - {}_a p_0 = 1 - e^{-\mu(a)}$ (this is why we put everything in terms of T), so

$$F_Y(y) = 1 - e^{-\mu(a)},$$

where $a = \frac{\ln(1-\delta y)}{-\delta}$. Therefore,

$$F_Y(y) = 1 - e^{\frac{\mu}{\delta}\ln(1-\delta y)} = 1 - e^{\ln(1-\delta y)^{\frac{\mu}{\delta}}} = 1 - (1-\delta y)^{\frac{\mu}{\delta}}$$

This is only defined for $y < \frac{1}{\delta}$.

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33. First we will calculate the value assuming we are standing at age 30, then we will accumulate for 40 years of interest and survivorship by dividing by ${}_{40}E_{30}$.

- Begin with an annuity paying 100 per month for life → $1200a_3^{(12)}0$.
- After a period of 10 years, add on an entirely new annuity paying 100 per month. This will produce payments of 200 per month from that time forward → $1200_{10|}a_{30}^{(12)}$
- After 20 years, add 300 per month to get to 500 per month total: → $3600_{20|}a_{30}^{(12)}$.
- After 30 years, add 500 per month to get to 1000 per month total: → $6000_{30|}a_{30}^{(12)}$.
- After 40 years, subtract 1000 per month to cancel out all payments currently being made.→ $-12000_{40|}a_{30}^{(12)}$.

Now divide the total of these five annuities by ${}_{40}E_{30}$ and factor out 1200 to get the answer in the text.

45.

(A) $\ddot{a}_x^{(m)} = \alpha(m)\ddot{a}_x - \beta(m)$. Substituting $m = 12$ and $x = 40$ results in

$$\ddot{a}_{40}^{(12)} = \alpha(12)\ddot{a}_{40} - \beta(12).$$

The table available on exam day shows that $\alpha(12) = 1.00028$ and $\beta(12) = 0.4612$. (If you need α and β on the exam, it will probably be for an m given in the table, but this is not for certain.)

$$\Rightarrow \ddot{a}_{40}^{(12)} = (1.00028)(14.817) - 0.46812 = 14.353.$$

(B) $\ddot{a}_{40:\overline{30}|}^{(12)} = \ddot{a}_{40}^{(12)} - {}_{30}E_{40}\ddot{a}_{70}^{(12)}$.

$$\ddot{a}_{70}^{(12)} = \alpha(12)\ddot{a}_{70} - \beta(12) = (1.00028)(8.569) - 0.46812 = 8.103,$$

$${}_{30}E_{40} = v^{30}\,{}_{30}p_{40} = \left(\frac{1}{1.06}\right)^{30}\frac{\ell_{70}}{\ell_{40}} = (0.1741)\left(\frac{66162}{93132}\right) = 0.1237$$

$$\ddot{a}_{40:\overline{30}|}^{(12)} = 14.353 - (0.1237)(8.103) = 13.351.$$

(C)

$$_{30|}\ddot{a}_{40}^{(12)} = {}_{30}E_{40}\ddot{a}_{70}^{(12)} = (0.1237)(8.103) = 1.0023$$

51.

(A) $30\ddot{a}_{65} + 20\ddot{a}_{75} + 10\ddot{a}_{85} = 30(9.897) + 20(7.217) + 10(4.698) = 488.23$.

(B) Since the lives are independent, the variance for the group is the sum of the individual variances.

$$\Rightarrow \text{Var} = 30\left(\frac{{}^2A_{65} - A_{65}^2}{d^2}\right) + 20\left(\frac{{}^2A_{75} - A_{75}^2}{d^2}\right) + 10\left(\frac{{}^2A_{85} - A_{85}^2}{d^2}\right)$$

$$= 30\left(\frac{0.2360 - [0.4398]^2}{[0.0566]^2}\right) + 20\left(\frac{0.3868 - [0.5915]^2}{[0.0566]^2}\right) + 10\left(\frac{0.5616 - [0.7341]^2}{[0.0566]^2}\right)$$

$$= 398.71 + 230.54 + 70.85 = 700.01$$

I assume the difference with the text is rounding.

(C) Although the text doesn't say so, the intent is for us to use the normal approximation. If X is the random variable for the total benefits paid on the portfolio, then $\mu_X = 488.23$ and $\sigma_X = 26.46$. Since $\Phi(1.645) = 0.95$, we are looking for the value of X such that

$$\frac{X - \mu_X}{\sigma_X} = 1.645,$$

$$\Rightarrow \frac{X - 488.23}{26.46} = 1.645,$$
$$\Rightarrow X = 1.645(26.46) + 488.23 = 531.76.$$

53.

$$\ddot{a}_{13:\overline{52}|} = \ddot{a}_{13} - {}_{52}E_{13}\, \ddot{a}_{65} = 16.813 - \left(\frac{1}{1.06}\right)^{52} \frac{\ell_{65}}{\ell_{13}} 9.897$$
$$= 16.813 - (0.0483)\frac{75340}{96808} 9.897 = 16.441$$

56.

$$\overline{a}_{13:\overline{52}|} = \overline{a}_{13} - {}_{52}E_{13}\overline{a}_{6}5$$

We saw in Exercise 53 that ${}_{52}E_{13} = 0.0376$. Also,

$$\overline{a}_{13} = \alpha(\infty)\ddot{a}_{13} - \beta(\infty).$$

From the distribution tables available at the exam we can see that $\alpha(\infty) = 1.00028$, and $\beta(\infty) = 0.50985$. So

$$\overline{a}_{13} = 1.00028(16.813) - 0.50985 = 16.308,$$
$$\overline{a}_{65} = 1.00028(9.897) - 0.50985 = 9.390.$$
$$\Rightarrow \overline{a}_{13:\overline{52}|} = 16.308 - (0.0376)(9.39) = 15.955.$$

Additional Problem:

(A) The cash flows are exactly the same – 1 at each year anniversary – except that $\ddot{a}_x$ includes a payment at time $t = 0$ and a_x does not. Therefore, $\ddot{a}_x = a_x + 1$.

(B) $\ddot{a}_{x:\overline{n}|}$ consists of cash flows at times $t = 0, t = 1, ..., t = n - 1$ if the annuitant is alive on those dates. $a_{x:\overline{n}|}$ has payments at times $t = 1, t = 2, ...t = n$, if the annuitant is alive on those dates. Only two of the payments are different, so $\ddot{a}_{x:\overline{n}|} - a_{x:\overline{n}|} = 1 - {}_nE_x$.

(C) This one is trickier: $a_{65:\overline{10}|}$ consists of payments at times $t = 1, ..., t = 10$ if the annuitant is alive on those dates. This is the same set of cash flows for a 1-year deferred, 10-year annuity due. Therefore,

$$a_{65:\overline{10}|} = {}_1E_{65}\, \ddot{a}_{66:\overline{10}|} = \frac{1}{1.08} p_{65}\, \ddot{a}_{66:\overline{10}|} = (0.9259)(0.95)(6) = 5.28.$$

(D) $a_{30:\overline{20}|} = {}_1E_{30}\, \ddot{a}_{31:\overline{20}|} = vp_{30}\, (\ddot{a}_{31} - {}_{20}E_{31}\, \ddot{a}_{51})$
$= (0.9434)(0.9985)\, [15.77 - v^{20}\, {}_{20}p_{31}\, \ddot{a}_{51}]$
$= (0.9420)\, [15.77 - 0.547(13.08)] = 11.01$

Actuarial Mathematics: Chapter 6 – Suggested Problems and Solutions.

Suggested Problems:

4, 10(long but good practice! Do the 1st and last rows only), 13, 16, 17, 27, 29, 31, 32

Suggested Solutions:

4.

(A) The fact that $T(x)$ has an exponential distribution tells us this is a constant force of mortality problem.

$$\Rightarrow \overline{P}(\overline{A}_x) = \mu.$$

Also $50 = \mathrm{E}[T(X)] = \frac{1}{\mu}$. Therefore $\overline{P}(\overline{A}_x) = 0.02$.

(B) We need to find the breakeven premium for the median future lifetime of (x). So first, we find t such that $\Pr[T > t = 0.5]$, or ${}_tp_0 = 0.5$.

$${}_tp_0 = e^{-\mu t} = e^{-0.02t} = 0.5$$

$$\Rightarrow t = \frac{\ln(0.5)}{-0.02} = 34.66.$$

Now write the loss function for this period of time and set equal to zero.

$$L = v^{34.66} - P \cdot \overline{a}_{\overline{34.66}|} = 0$$

$$\Rightarrow P \cdot \frac{1 - v^{34.66}}{\delta} = v^{34.66}, \quad \Rightarrow P \cdot \frac{1 - e^{-(0.06)34.66}}{0.06} = e^{-(0.06)34.66}$$

$$\Rightarrow P(14.583) = 0.125 \qquad \Rightarrow P = 0.00857$$

(C) If $\delta = 0$, then $\overline{a}_{\overline{34.66}|} = 34.66$ and $v = 1$.

$$\Rightarrow P = \frac{1}{34.66} = 0.02885$$

10. I will do 2 of these, the answers in the text should be enough for the rest. You need to assume UDD to do the non-fully-discrete premiums.

$$P_{35:\overline{10}|} = \frac{A_{35:\overline{10}|}}{\ddot{a}_{35:\overline{10}|}} = \frac{A^{\;1}_{35:\overline{10}|} + {}_{10}E_{35}}{\ddot{a}_{35} - {}_{10}E_{35}\,\ddot{a}_{45}} = \frac{A_{35} - {}_{10}E_{35}\,A_{45} + {}_{10}E_{35}}{\ddot{a}_{35} - {}_{10}E_{35}\,\ddot{a}_{45}}$$

$$= \frac{(0.1287) - (0.543)(0.2012) + (0.543)}{15.39 - (0.543)(14.11)} = 0.0728.$$

$$\overline{P}\left(\overline{A}^{\,1}_{35:\overline{10|}}\right) = \frac{\overline{A}^{\,1}_{35:\overline{10|}}}{\overline{a}_{35:\overline{10|}}} = \frac{\overline{A}_{35} - {}_{10}E_{35}\,\overline{A}_{45}}{\overline{a}_{35} - {}_{10}E_{35}\,\overline{a}_{45}}$$

$$= \frac{\frac{i}{\delta}A_{35} - {}_{10}E_{35}\,\frac{i}{\delta}A_{45}}{\alpha(\infty)\ddot{a}_{35} - \beta(\infty) - {}_{10}E_{35}\left[\alpha(\infty)\ddot{a}_{45} - \beta(\infty)\right]}$$

$$\frac{(1.0297)(0.1287) - (0.543)(1.0297)(0.2012)}{(1.00028)(15.39) - (0.5099) - (0.543)\left[(1.00028)(14.11) - (0.5099)\right]}$$

$$= 0.00267.$$

13.

$$P^{(2)}_{50} = \frac{A_{50}}{\ddot{a}^{(2)}_{50}} = \frac{0.249}{\alpha(2)\ddot{a}_{50} - \beta(2)} = \frac{0.249}{(1.000212)(13.27) - 0.2574} = 0.0191$$

16.

$$P^{(12)}_{\overset{1}{x}:\overline{20|}} = \frac{A^{\,1}_{x:\overline{20|}}}{\ddot{a}^{(12)}_{x:\overline{20|}}} \qquad P_{\overset{1}{x}:\overline{20|}} = \frac{A^{\,1}_{x:\overline{20|}}}{\ddot{a}_{x:\overline{20|}}}$$

$$\Rightarrow \frac{P^{(12)}_{\overset{1}{x}:\overline{20|}}}{P_{\overset{1}{x}:\overline{20|}}} = \frac{\ddot{a}_{x:\overline{20|}}}{\ddot{a}^{(12)}_{x:\overline{20|}}} = 1.032$$

Similar analysis shows that

$$\frac{P^{(12)}_{x:\overline{20|}}}{P_{x:\overline{20|}}} = \frac{\ddot{a}_{x:\overline{20|}}}{\ddot{a}^{(12)}_{x:\overline{20|}}} = 1.032$$

$$\Rightarrow P^{(12)}_{x:\overline{20|}} = (1.032)(0.04) = 0.0413.$$

---

17. See text for solution. The idea is that the more often you pay per year, the more premium the insurance company forgoes when you die, so the premium must be higher if you pay more often. For example, if your annual premium is 1000 per year and you decide to go to paying 500 per half-year. If you die in March, the insurance company is out 500. Your half-year premium has to be higher to make up for this. (And I thought they were just punishing us for the administrative hassle!)

---

27. A 3-premium problem.

$$\frac{{}_nP_x - P^{\;1}_{x:\overline{n}|}}{P_{x:\overset{1}{\overline{n}|}}} = A_{x+n}$$

$$\Rightarrow \frac{{}_{15}P_{45} - P^{\;1}_{45:\overline{15}|}}{P_{45:\overset{1}{\overline{15}|}}} = A_{60}$$

$$\Rightarrow 0.038 - P^{\;1}_{45:\overline{15}|} = (0.625)P_{45:\overset{1}{\overline{15}|}} = (0.625)\left(P_{45:\overline{15}|} - P^{\;1}_{45:\overline{15}|}\right)$$

$$\Rightarrow 0.038 - P^{\;1}_{45:\overline{15}|} = 0.035 - 0.625P^{\;1}_{45:\overline{15}|}$$

$$\Rightarrow P^{\;1}_{45:\overline{15}|} = 0.008$$

29.

$$P\ddot{a}_{25:\overline{40}|} + P \cdot {}_{10}E_{25} \cdot \ddot{a}_{35:\overline{30}|} = A_{25} + {}_{10}E_{25}A_{35}$$

$$\Rightarrow P = \frac{A_{25} + {}_{10}E_{25}A_{35}}{\ddot{a}_{25:\overline{40}|} + \cdot{}_{10}E_{25} \cdot \ddot{a}_{35:\overline{30}|}}$$

The solution in the text is equivalent.

31.

(A)

$$\mathrm{E}[L] = A_x - P\ddot{a}_x = 0.4 - (0.048)\left(\frac{1-0.4}{0.06}\right) = -0.08$$

So there is an expected profit of 0.08 on the unit policy.

(B)

$$\mathrm{Var}[L] = \left(1 + \frac{P}{d}\right)^2 \left[{}^2A_x - (A_x)^2\right] = (3.24)(0.04) = 0.1296$$

(C) The expected loss on all 100 policies will be

$$80(-0.08) + 20(4)(-0.08) = 12.8$$

The variance will be

$$80(0.1296) + 20(16)(0.1296) = 51.84$$

so the standard deviation for the block is $\sigma = 7.2$. Therefore

$$\Pr[L > 20] = \Pr\left[\frac{L-\mu}{\sigma} > \frac{20-12.8}{7.2}\right] = \Pr\left[\frac{L-\mu}{\sigma} > 1\right]$$

$$= 1 - \Phi(1) = 0.1587.$$

32.

$$P\ddot{a}_x + {}_{10}E_x P\ddot{a}_{x+5} = \overline{A}_x$$
$$\Rightarrow P = \frac{\overline{A}_x}{\ddot{a}_x + {}_{10}E_x\, \ddot{a}_{x+5}}$$

The solution in the text is equivalent.

Actuarial Mathematics: Chapter 7 – Suggested Problems and Solutions.

Suggested Problems:

8, 14, 16, 19, 23(Middle row and last column), 24, 26, 29, 30, 31

Suggested Solutions:

The text solutions are adequate for 8, 14, 16, 19.

23. You have to assume UDD to do most of these. On the test, do not assume UDD unless they give it to you! I will do ${}_{10}\overline{V}\left(\overline{A}_{35}\right)$ and ${}_{10}V^{1}_{35:\overline{30|}}$. The numerical answers for the others are in the text.

$$ {}_{10}\overline{V}\left(\overline{A}_{35}\right) = \overline{A}_{45} - \overline{P}\left(\overline{A}_{35}\right)\overline{a}_{45} = \overline{A}_{45} - \frac{\overline{A}_{35}}{\overline{a}_{35}}\overline{a}_{45} $$

$$ \overline{A}_{35} = \frac{i}{\delta}A_{35} = (1.02971)(0.12872) = 0.13254, $$

$$ \overline{A}_{45} = \frac{i}{\delta}A_{45} = (1.02971)(0.2012) = 0.20718, $$

$$ \overline{a}_{35} = \frac{1-\overline{A}_{35}}{\delta} = 14.887, $$

$$ \overline{a}_{45} = \frac{1-\overline{A}_{45}}{\delta} = 13.606. $$

$$ \rightarrow {}_{10}\overline{V}\left(\overline{A}_{35}\right) = 0.20718 - \frac{0.13254}{14.887}(13.606) = 0.0860 $$

For a change of pace, I will do the second one retrospecively.

$$ {}_{10}V^{1}_{35:\overline{30|}} = P^{1}_{35:\overline{30|}}\ddot{s}_{35:\overline{10|}} - {}_{10}k_{35} = \frac{A^{1}_{35:\overline{30|}}}{\ddot{a}_{35:\overline{30|}}}\frac{\ddot{a}_{35:\overline{10|}}}{{}_{10}E_{35}} - \frac{A^{1}_{35:\overline{10|}}}{{}_{10}E_{35}} $$

$$ {}_{10}E_{35} = (0.5432) $$

$$ A^{1}_{35:\overline{30|}} = A_{35} - {}_{30}E_{35}\,A_{65} = (0.12872) - (0.13924)(0.4398) = 0.0675 $$

$$ A^{1}_{35:\overline{10|}} = A_{35} - {}_{10}E_{35}\,A_{45} = (0.12872) - (0.5432)(0.2012) = 0.01943 $$

$$ \ddot{a}_{35:\overline{10|}} = \ddot{a}_{35} - {}_{10}E_{35}\ddot{a}_{45} = 15.393 - (0.5432)(14.112) = 7.727 $$

$$ \ddot{a}_{35:\overline{30|}} = \ddot{a}_{35} - {}_{30}E_{35}\ddot{a}_{65} = 15.393 - (0.13924)(9.897) = 14.015 $$

$$\rightarrow {}_{10}V^{1}_{35:\overline{30|}} = \frac{(0.0675)(7.727)}{(14.015)(0.5432)} - \frac{0.01943}{0.5432} = 0.03273$$

24.

(A) This one is not true - see equation 7.5.2 in the text (remove the h in the equation).

(B)

$$\begin{aligned} {}_kV\left(\overline{A}_x\right) &= \overline{A}_{x+k} - P(\overline{A}_x)\ddot{a}_{x+k} = \frac{i}{\delta}A_{x+k} - \frac{\frac{i}{\delta}A_x}{\ddot{a}_x}\ddot{a}_{x+k} \\ &= \frac{i}{\delta}\left[A_{x+k} - \frac{A_x}{\ddot{a}_x}\ddot{a}_{x+k}\right] = \frac{i}{\delta}\,{}_kV_x \end{aligned}$$

(C) Same analysis as for part b).

---

26. I am a little sorry I recommended this one since it hinges on a detail that might not be useful on test day. Basically, for parts a, b, and c, the algebra works out just as for continous formulas done in the manual. For part d, the algebra breaks down because $P^{(m)}_{40}$ is defined by

$$P^{(m)}_{40} = \frac{A_{40}}{\ddot{a}^{(m)}_{40}} \neq \frac{A^{(m)}_{40}}{\ddot{a}^{(m)}_{40}}.$$

The two are not equal because A_{40} pays at the end of year of death and $A^{(m)}_{40}$ pays at the end of the m-th of a year of death.

This is not a super-useful fact and you should feel comfortable NOT working out this question. Hopefully the dicussion above will help you remember the correct definition of $P^{(m)}_{40}$!

---

29. This is a 3-premium problem. Note that paying P_x for n years provides term protection for n years AND a reserve ${}_nV_x$ at the end of n years. The premium $P^{1}_{x:\overline{n|}}$ provides only the term protection for the n years and no reserve at the end. So

$$\frac{P_x - P^{1}_{x:\overline{n|}}}{P_{x:\overline{n|}}^{\;\;1}} = {}_nV_x$$

$$\Rightarrow \frac{0.024 - P^{1}_{x:\overline{n|}}}{0.2} = 0.08$$

$$\Rightarrow P^{1}_{x:\overline{n|}} = 0.008.$$

---

30. This is a good example of how "quick and dirty" the annuity reserve formula can be for tying together different reserves where beginning and ending years are overlapping.

$$ {}_{10}V_{35} = 1 - \frac{\ddot{a}_{45}}{\ddot{a}_{35}}; \qquad {}_{20}V_{35} = 1 - \frac{\ddot{a}_{55}}{\ddot{a}_{35}}; \qquad {}_{10}V_{45} = 1 - \frac{\ddot{a}_{55}}{\ddot{a}_{45}} $$

Using the numbers we are given:

$$ \frac{\ddot{a}_{45}}{\ddot{a}_{35}} = 0.85; \qquad \frac{\ddot{a}_{55}}{\ddot{a}_{35}} = 0.646 $$

$$ \Rightarrow \frac{\ddot{a}_{55}}{\ddot{a}_{45}} = \frac{0.646}{0.85} = 0.76 \qquad \Rightarrow {}_{10}V_{45} = 0.24. $$

---

31. a) We need to solve the following equation for P:

$$ P_{25}\, \ddot{a}_{25:\overline{10}|} + P \cdot {}_{10}E_{25}\, \ddot{a}_{35:\overline{30}|} = A_{25} $$

Using the ILT

$$ P_{25} = \frac{A_{25}}{\ddot{a}_{25}} = \frac{0.0817}{16.224} = 0.00503 $$

$$ {}_{10}E_{25} = v^{10}\, {}_{10}p_{25} = \left(\frac{1}{1.06}\right)^{10} \frac{94{,}207}{95{,}650} = 0.5496 $$

$$ \ddot{a}_{25:\overline{10}|} = \ddot{a}_{25} - {}_{10}E_{25}\, \ddot{a}_{35} = 16.224 - (0.5496)(15.393) = 7.764 $$

$$ \ddot{a}_{35:\overline{30}|} = \ddot{a}_{35} - {}_{30}E_{35}\, \ddot{a}_{65} = 15.393 - \left(\frac{1}{1.06}\right)^{30} \frac{\ell_{65}}{\ell_{35}}(9.897) = 14.021 $$

Using the equation we started with:

$$ (0.00503)(7.764) + P \cdot (0.5496)(14.021) = 0.0817 $$

$$ \Rightarrow P = 0.00553 $$

b)

$$ {}_{10}V = A_{35} - P\, \ddot{a}_{35:\overline{30}|} = 0.1287 - (0.00553)(14.021) = 0.051 $$

c) The value of the reserve plus the value of future premiums should equal the value of future benefits.

$$ {}_{10}V + P_{25}\, \ddot{a}_{35:\overline{30}|} = B \cdot A_{35} $$

$$ 0.051 + (0.00503)(14.021) = B \cdot (0.1287) $$

$$\Rightarrow B = 0.944(\text{slightly different than text answer.})$$

d)

$$ {}_{20}V = B \cdot A_{45} - P_{25}\, \ddot{a}_{45:\overline{20}|} = (0.944)(0.2012) - (0.00503)\ddot{a}_{45:\overline{20}|}$$

$$\ddot{a}_{45:\overline{20}|} = \ddot{a}_{45} - {}_{20}E_{45}\, \ddot{a}_{65} = 14.112 - (0.256)(9.897) = 11.575$$

$$\Rightarrow {}_{20}V = 0.132$$

Actuarial Mathematics: Chapter 8 – Suggested Problems and Solutions.

Suggested Problems:

14, 18abc, 30abc, 35ab

Suggested Solutions:

14. Those are not parentheses around the exponent they are brackets! That means this question involves apportionable premiums. Please disregard it!

---

18. a)

$$_{20}P_{25} = \frac{\ddot{a}_{25:\overline{20|}}}{A_{25}}$$

$$A_{25} = 0.0816$$

$$\ddot{a}_{25:\overline{20|}} = \ddot{a}_{25} - {}_{20}E_{25}\,\ddot{a}_{45} = 16.224 - (0.2987)(14.112) = 12.009$$

$$\Rightarrow {}_{20}P_{25} = \frac{0.0816}{12.009} = 0.00680$$

b) ${}^{20}_{19}V_{25} = (PVFB) - (PVFP)$, but there is only one premium left, so this equals

$$A_{44} - (0.0068) = 0.1926 - 0.0068 = 0.1858$$

c) ${}^{20}_{20}V_{25} = (PVFB) - (PVFP) = A_{45} - 0 = 0.2012$

d) Disregard!

---

30. a) The first premium, with interest, must provide death benefits for those who die in year 1 plus the reserve for those who live. Therefore

$$P(1+i) = 3q_x + p_x\,({}_1V)$$

$$(0.94)(1.2) = 3q_x + (1-q_x)(0.66) \qquad \Rightarrow q_x = 0.2$$

b)

$$({}_1V + P)(1+i) = 3q_{x+1} + p_{x+1}\cdot {}_2V$$

$$1.92 = 3q_{x+1} + (1-q_{x+1})(1.56) \qquad \Rightarrow q_{x+1} = 0.25$$

c) We'll do this the old-fashioned way, a la course 1.

Year of Death	${}_0L$	Prob
1	$-0.94 + \frac{3}{1.2} = 1.56$	0.2
2	$-0.94 - \frac{0.94}{1.2} + \frac{3}{(1.2)^2} = 0.360$	$(0.8)(0.25) = 0.2$
3+	$-0.94 - \frac{0.94}{1.2} - \frac{0.94}{(1.2)^2} + \frac{3}{(1.2)^3} = -0.640$	0.6

We already know that $\mathrm{E}[{}_0L] = 0$ since we are told that 0.94 is the benefit premium. Checking that this is true lets us know that we probably have the right values of ${}_0L$.

$$\mathrm{Var}[{}_0L] = \mathrm{E}\left[({}_0L)^2\right] - \left(\mathrm{E}\left[{}_0L\right]\right)^2$$

$$= (0.2)(1.56)^2 + 0.2(0.36)^2 + 0.6(0.64)^2 = 0.4867 + 0.0259 + 0.2458 = 0.7584$$

---

35. a) The single premium must provide for all of the future benefits:

$$S = 100{,}000 \cdot {}_{30}E_{35} + S \cdot A_{\overset{1}{35}:\overline{30|}}$$

$$S = \frac{100{,}000 \cdot {}_{30}E_{35}}{1 - A_{\overset{1}{35}:\overline{30|}}}$$

b)

$$\begin{aligned} {}_kV &= (PVFB) - (PVFP) = (PVFB) - (0) \\ &= 100{,}000 \cdot {}_{30-k}E_{35+k} + S \cdot A_{\overset{1}{35+k}:\overline{30-k|}} \end{aligned}$$

Actuarial Mathematics: Chapter 9 – Suggested Problems and Solutions.

Suggested Problems:

4(assume the lives are independent), 10, 12, 13, 22, 24, 25, 26, 28, 37

Suggested Solutions:

4. Text solution is sufficient.

---

10. (Note that this solution assumes the lives are independent.) The probability we are asked to find is equal to

$$\text{(The probability that } x \text{ dies in year } n) + \text{(The probability that } y \text{ dies in year } n)$$
$$-\text{(The probability that both die in year } n)$$

$$= {}_{n|}q_x + {}_{n|}q_y - {}_{n|}q_x \cdot {}_{n|}q_y.$$

This is not the same as ${}_{n|}q_{\overline{xy}}$ since both must be dead by the end of year n for ${}_{n|}q_{\overline{xy}}$. For the situation described in the problem, one or both may be dead at the end of year n.

---

12. (For this problem to work, you have to assume the lives are independent.)

$$0.2 = {}_{25}P_{25:50} = {}_{25}P_{25} \cdot {}_{25}P_{50}$$

$${}_{25}P_{25} = {}_{15}P_{25} \cdot {}_{10}P_{40}$$

$$\Rightarrow 0.2 = (0.9)_{10}P_{40} \cdot {}_{25}P_{50} = (0.9)_{35}P_{40}$$
$$\Rightarrow {}_{35}P_{40} = 0.2222$$

---

13. a) This is DeMoivre's law with $\omega = 100$

$${}_{10}P_{40:50} = {}_{10}P_{40} \cdot {}_{10}P_{50}$$

For DeMoivre, ${}_tp_x = \frac{\omega-x-t}{\omega-x}$, so

$${}_{10}P_{40:50} = \frac{50}{60} \cdot \frac{40}{50} = \frac{2}{3}$$

b)

$$_{10}P_{\overline{40:50}} = {}_{10}P_{40} + {}_{10}P_{50} - {}_{10}P_{40} \cdot {}_{10}P_{50} = \frac{50}{60} + \frac{40}{50} - \frac{2}{3} = 0.967$$

c) The text question is a typo - their answer reveals that they intended to ask for the much easier to determine $\overset{\circ}{e}_{40:50}$, but we can use an old trick from Algebra II to find $e_{40:50}$.

$$e_{xy} = \sum_{t=1}^{50} {}_tp_{xy} = \sum_{t=1}^{50} \frac{60-t}{60} \cdot \frac{50-t}{50}$$

$$= \sum_{t=1}^{50} \left(1 - \frac{t}{60}\right) \cdot \left(1 - \frac{t}{50}\right) = \sum_{t=1}^{50} \left(1 - 0.03667t + \frac{t^2}{3000}\right)$$

The first sum is just 50. The second and third sums can be done using

$$\sum_{t=1}^{n} t = \frac{n(n+1)}{2} \qquad \sum_{t=1}^{n} t^2 = \frac{n(n+1)(2n+1)}{6}$$

You probably won't need to be able to do the sum of squares on the exam.

$$\Rightarrow e_{xy} = 50 - 0.03667(1275) + \frac{1}{3000}(42925) = 17.56$$

d) $e_{\overline{xy}} = e_x + e_y - e_{xy}$. For DeMoivre's Law,

$$e_x = \frac{\omega - x - 1}{2}$$

$$\Rightarrow e_{40} = 29.5 \qquad e_{50} = 24.5$$

$$\Rightarrow e_{\overline{40:50}} = 29.5 + 24.5 - 17.56 = 36.44$$

e) First we need the $\overset{\circ}{e}_{40:50}$ they meant to ask for in the previous part.

$$\overset{\circ}{e}_{xy} = \int_0^{50} {}_tp_x \cdot {}_tp_y dt$$

The integral stops at $t = 50$ since (50) will certainly be dead by then.

$$\overset{\circ}{e}_{40:50} = \int_0^{50} {}_tp_{40} \cdot {}_tp_{50}\, dt = \int_0^{50} \left(\frac{60-t}{60}\right)\left(\frac{50-t}{50}\right) dt$$

$$= \frac{1}{3000} \int_0^{50} (t^2 - 110t + 3000)\, dt = 18.06$$

Now,

$$\mathrm{Var}[T(40:50)] = 2\int_0^{50} t \cdot {}_tp_{xy}\, dt - \left(\overset{\circ}{e}_{xy}\right)^2$$

$$= \frac{2}{3000}\int_0^{50}(t^3 - 110t^2 + 3000t)\,dt - (18.06)^2$$
$$= \frac{1}{1500}\left[\frac{t^4}{4} - \frac{110t^3}{3} + 1500t^2\right]_0^{50} - 326.16$$
$$= 486.11 - 326.16 = 160$$

f) For this one, we need $\mathring{e}_{\overline{40:50}} = 36.94$.

$$\operatorname{Var}[T\left(\overline{40:50}\right)] = 2\int_0^{50} t\cdot {}_tp_{\overline{xy}}\,dt - \left(\mathring{e}_{\overline{xy}}\right)^2$$

$$= 2\int_0^{50} t\cdot\left({}_tp_{40} + {}_tp_{50} - {}_tp_{40:50}\right)\,dt - (36.94)^2$$

$$= 2\int_0^{50} t\cdot\frac{60-t}{60}\,dt + 2\int_0^{50} t\cdot\frac{60-t}{60}\,dt - 2\int_0^{50} t\cdot\frac{60-t}{60}\cdot\frac{50-t}{50}\,dt - (36.94)^2$$

The third integral was done in Part e). Changing the other two to polynomials and integrating results in the book's answer — 182.

g)

$$\operatorname{Cov}\left[T(40:50), T\left(\overline{40:50}\right)\right] = \left(\mathring{e}_{40} - \mathring{e}_{40:50}\right)\left(\mathring{e}_{50} - \mathring{e}_{40:50}\right)$$
$$= (30 - 18.06)(25 - 18.06) = 11.94\cdot 6.94 = 82.9$$

h) The correlation coefficient ρ is given by

$$\rho = \frac{\operatorname{Cov}\left[T(40:50), T\left(\overline{40:50}\right)\right]}{\sqrt{\operatorname{Var}[T(40:50)]\cdot\operatorname{Var}[T\left(\overline{40:50}\right)]}} = \frac{82.9}{\sqrt{(160.11)(182.33)}} = 0.49$$

---

22. $\overline{A}_{\overline{x:\overline{n}|}}$ refers to a benefit that pays 1 at the moment of death or at the end of n years, whichever is later. We want to show that this benefit is the same as that paid by a combination of insurances with present value equal to

$$\overline{A}_x - \overline{A}_{x:\overline{n}|} + v^n.$$

This is a whole life policy minus an n-year endowment plus a payment at time n regardless of what happens. If (x) dies in the first n years, the first two insurances cancel out and there will be a payment at time n. If (x) dies after time n, the endowment and the fixed payment cancel out and (x) receives 1 at the time of death. This is that same benefit structure as the one with APV equal to $\overline{A}_{\overline{x:\overline{n}|}}$. (This example is not completely exam-like, but this kind of logic could be useful on the test!)

---

24. $\overline{a}_{25:\overline{25}|}$ pays while (25) is alive and below age 50, while $\overline{a}_{30:\overline{20}|}$ pays while (30) is alive and below age 50. Finally, $\overline{a}_{25:30:\overline{20}|}$ pays while *both* (25) and (30) are alive and below 50. So the APV of an annuity that pays while either (25) or (30) is alive and below 50 is given by

$$\overline{a}_{25:\overline{25}|} + \overline{a}_{30:\overline{20}|} - \overline{a}_{25:30:\overline{20}|}.$$

---

25. This uses the same logic as Problem 24. The answer is the annuity for (35) plus the annuity for (30) minus the annuity for the joint life survival status as givenin the text.

---

26. This annuity will pay

- 1/2 to y as long as y is alive up to n years, plus
- 1/3 to x as long as x is alive up to n years , plus
- as long as both are alive and $t < n$, it will pay 1 = (1/2 + 1/3 +1/6).

So this annuity is the same as one having APV equal to

$$\frac{1}{2}\ddot{a}_{y:\overline{n}|} + \frac{1}{3}\ddot{a}_{x:\overline{n}|} + \frac{1}{6}\ddot{a}_{x:y:\overline{n}|}.$$

---

28. This one is messy. Here is how I think of it: The annuity

- pays 1 to (55) if alive after age 60 but not in years 6-15 if both are alive. And
- pays 1 to (40) if alive after age 60 but not if both are alive (since in that case the payment is already being made - to (55).

The APV of the first annuity is

$$_{5|}\overline{a}_{55} - {}_{5|10}\overline{a}_{40:55},$$

and the APV of the second is

$$_{20|}\overline{a}_{40} - {}_{20|}\overline{a}_{40:55}.$$

Summing these results gives the answer in the text.

---

37. Under UDD, ${}_tp_x = 1 - t \cdot q_x$, so ${}_tp_x = {}_tp_y = 1 - t$.

$$\mathring{e}_{xy} = \int_0^1 {}_tp_{xy}\,dt = \int_0^1 {}_tp_x \cdot {}_tp_y\,dt = \int_0^1 (1-t)^2 dt = \int_0^1 (1 - 2t + t^2) dt = \frac{1}{3}$$

Actuarial Mathematics: Chapter 10 – Suggested Problems and Solutions.

Suggested Problems:

1, 2, 4, 5, 7, 8, 10, 14, 17, 28, 29

Suggested Solutions:

1. All forces of decrement are constant so $\mu_x^{(\tau)} = \mu_x^{(1)} + \cdot + \mu_x^{(m)}$ is constant.

a) $f_{T,J}(t,j) =$ (prob of surviving to time t) $\cdot$ (prob of dying at t due to cause j)

$$= {}_tp_x^{(\tau)} \cdot \mu_x^{(j)}(t) = e^{-t\mu_x^{(\tau)}} \cdot \mu_x^{(j)}$$

b) As in Section 10.2 of the manual,

$$f_J(j) = \frac{\mu_x^{(j)}}{-\mu_x^{(\tau)}}.$$

c) $f_T(t) = {}_tp_x^{(\tau)} \cdot \mu_x^{(\tau)} = e^{-t\mu_x^{(\tau)}} \cdot \mu_x^{(\tau)}$

---

2. a)$f_{T,J}(t,j) = {}_tp_x^{(\tau)} \cdot \mu_x^{(j)}(t)$

$${}_tp_{50}^{(\tau)} = e^{-\int_0^t \left(\mu_{50}^{(1)}(s)+\mu_{50}^{(2)}(s)\right)ds} = e^{-\int_0^t \frac{3}{50-s}ds} = e^{3\ln(50-s)|_0^t} = \left(\frac{50-t}{50}\right)^3$$

$$f_{T,J}(t,1) = \left(\frac{50-t}{50}\right)^3 \cdot \frac{1}{50-t} = \frac{(50-t)^2}{50^3}$$

$$f_{T,J}(t,2) = \left(\frac{50-t}{50}\right)^3 = \frac{2(50-t)^2}{50^3}$$

b)

$$f_T(t) = {}_tp_x^{(\tau)} \cdot \mu_x^{(\tau)}(t) = \left(\frac{50-t}{50}\right)^3 \cdot \frac{3}{50-t} = \frac{3}{50^3} \cdot (50-t)^2$$

c) Note that both forces of decrement blow up at $t = 50$, meaning that decrement by $t = 50$ is certain.

$$f_J(j) = {}_\infty q_x^{(j)} = {}_{50}q_x^{(j)}$$

$$f_J(1) = \int_0^{50} {}_tp_x^{(\tau)} \cdot \mu_x^{(1)}(t)\, dt = \int_0^{50} \frac{(50-t)^2}{50^3}\, dt = \left. \frac{-(50-t)^3}{3 \cdot 50^3} \right|_0^{50} = \frac{1}{3}$$

So $f_J(1) = \frac{1}{3}$, and this implies $f_J(2) = \frac{2}{3}$.

d)

$$\frac{\mu_x^{(j)}(t)}{\mu_x^{(\tau)}(t)} = \begin{cases} \frac{1}{3}, & j = 1 \\ \frac{2}{3}, & j = 2 \end{cases}$$

---

4. a)

$${}_3p_{65}^{(\tau)} = \frac{\ell_{68}}{\ell_{65}} = 0.75321$$

b)

$${}_{3|}q_{65}^{(1)} = {}_3p_{65}^{(\tau)} \cdot q_{68}^{(1)} = (0.75321)(0.05) = 0.03766$$

c)

$${}_3q_{65}^{(2)} = \frac{d_{65}^{(2)} + d_{66}^{(2)} + d_{67}^{(2)}}{\ell_{65}} = \frac{50 + 55.8 + 59.24}{1000} = 0.16504$$

---

5. a) Find the probability that a single person will graduate and multiply by 1000.

$${}_tp_x^{(\tau)} = (0.6)(0.7)(0.8)(0.9) = 0.3024$$

$$\mathrm{E}[N] = 302.4$$

For the variance, assuming the lives are independent, we can find the variance for one person and then multiply by 1000.

$$\mathrm{Var} = p(1-p) = (0.3024)(0.6976) = 0.21095$$

$$1000(p)(1-p) = 210.95$$

b) i) Prob of failure = 0.15 + 0.1(0.6) + 0.5(0.6)(0.7) = 0.231
ii) 1000(0.231)(0.769) = 177.64

---

7. a) $\ell_x^{(\tau)} = {}_tp_0^{\tau)} \cdot \ell_0$

$${}_tp_0^{\tau)} = e^{-\int_0^t \left(\frac{1}{a-s}+1\right) ds} = e^{-\int_0^t \frac{1}{a-s}\, ds} \cdot e^{-\int_0^t ds} = e^{-t} \cdot \frac{a-t}{a}$$

$$\ell_0 = a \qquad \Rightarrow \ell_x^{(\tau)} = (a-x)e^{-x}$$

b)

$$d_x^{(1)} = \ell_x \cdot \int_0^1 {}_tp_x^{(\tau)} \cdot \frac{1}{a-x-t}\,dt$$

$${}_tp_x^{(\tau)} = {}_tp_x^{\prime(1)} \cdot {}_tp_x^{\prime(2)} = \frac{a-x-t}{a-x} \cdot e^{-t}$$

$$d_x^{(1)} = (a-x)e^{-x} \cdot \int_0^1 \frac{e^{-t}}{a-x} = e^{-x}(1-e^{-1})$$

c) This can be done two ways. First the straight-ahead method:

$$d_x^{(2)} = (a-x)e^{-x} \int_0^1 \frac{a-x-t}{a-x\,e^{-t}} \cdot 1\,dt = e^{-x} \int_0^1 (a-x-t)\,e^{-t}\,dt$$

This integral can be done using integration by parts to get the answer in the book. Or you can notice that $d_x^{(1)} + d_x^{(2)} = d_x^{(\tau)}$ and use the fact that

$$d_x^{(\tau)} = \ell_x - l_{x+1} = (a-x)e^{-x} - (a-x-1)(e^{-x} \cdot e^{-1})$$

$$\Rightarrow d_x^{(2)} = d_x^{(\tau)} - d_x^{(1)} = (a-x)e^{-x} - (a-x-1)e^{-x-1} - e^{-x}(1-e^{-1})$$

This mess is also equal to the answer in the text.

8.

$$\ell_x^{(\tau)} = 1000 \cdot {}_tp_0^{(\tau)}$$

$${}_tp_0^{(\tau)} = e^{-\int_0^x \frac{2t}{a-t^2}} \cdot e^{-cx}$$

u-substitution: let $u = a - t^2$; $du = -2t\,dt$. Then the integral equals

$$e^{\int_0^t \frac{du}{u}} \cdot e^{-cx} = e^{\ln u} \cdot e^{-cx} = e^{\ln(a-t^2)\big|_0^x} \cdot e^{-cx} = \frac{a-x^2}{a} \cdot e^{-cx}$$

So the answer is

$$1000\frac{a-x^2}{a} \cdot e^{-cx}$$

10.

$$q_0^{\prime(1)} = 1 - p_0^{\prime(1)} = 1 - \left(P_0^{(\tau)}\right)^{\frac{q_0^{(1)}}{q_0^{(\tau)}}} = 1 - (0.6)^{\frac{3}{8}} = 0.174$$

The rest of the table is done the same way.

14. $q_{40}^{(\tau)} = 1 - p_x^{(\tau)} = 1 - \left(p_x^{'(1)}\right)\left(p_x^{'(2)}\right) = 1 - 0.9408 = 0.0592$

---

17. $p_x^{(\tau)} = p_x^{'(1)} \cdot p_x^{'(2)} \cdot p_x^{'(3)}$

$$\Rightarrow q_{62}^{(\tau)} = 1 - p_{62}^{(\tau)} = 1 - (0.98)(0.97)(0.8) = 0.2395$$

$$\frac{q_{62}^{(1)}}{q_{62}^{(\tau)}} = \frac{\ln p_x^{'(1)}}{\ln p_x^{(\tau)}} = \frac{\ln(0.98)}{\ln(0.7605)} = 0.0738$$

$$\Rightarrow q_{62}^{(1)} = (0.0738)(0.2395) = 0.0177$$

The same approach works for the rest of the table. See the text for the complete table.

---

28. $q_{69}^{'(3)} = 1$

$$q_{69}^{(3)} = 1 - q_{69}^{(1)} - q_{69}^{(2)} = 1 - 0.01843 - 0.03723 = 0.9443$$

---

29. A very exam-like question!

All of the withdrawalss occur first so really these two decrements act independently of each other. $d_{50}^{(2)} = 1000 \cdot q_{50}^{(2)} = 200$. This implies that $d_{50}^{(1)} = 12$ So, just after the year begins, 800 people are left that are subject only to decrement 1 during the year. So $q^{'(50)}$ must act on the 800 remaining people to produce 12 deaths during the year.

$$\Rightarrow q^{'(50)} = \frac{12}{800} = 0.015$$

Actuarial Mathematics: Chapter 11 – Suggested Problems and Solutions.

Suggested Problems:

1, 2

Suggested Solutions:

1.

$$\text{APV(Death Ben)} = 20{,}000 \int_0^{40} v^t \; {}_tp_{30}^{(\tau)} \; \mu_{30}^{(1)}(t)\, dt$$

$$\text{APV(Survival Ben)} = 12{,}000 v^{40} \; {}_{40}p_{30}^{(\tau)} \; \overline{a}_{70}$$

$$\text{APV(Withdrawal Ben)} = 300 \int_0^{40} v^t \; {}_tp_{30}^{(\tau)} \mu_{30}^{(2)}(t) \; (t) \; {}_{40-t|}\overline{a}_{30+t}\, dt$$

Adding them together gives the APV of all three benefits combined.

---

2. a) The expected value is equal to

$$\overline{A}^{\,1}_{x:\overline{20}|} + \overline{A}^{\,acc.\,1}_{x:\overline{20}|}$$

where $\overline{A}^{\,1}_{x:\overline{20}|}$ is a regular 20-year term policy and $\overline{A}^{\,acc.\,1}_{x:\overline{20}|}$ is a term policy for accidental death only. The regular term policy is not too bad since everything is constant force of mortality:

$$\overline{A}_x = \frac{\mu^{(\tau)}}{\mu^{(\tau)} + \delta} \qquad \text{(for any } x\text{)}$$

$${}_{20}E_x = e^{-20\delta} \cdot e^{-20\mu^{(\tau)}}$$

$$\overline{A}^{\,1}_{x:\overline{20}|} = \overline{A}_x - {}_{20}E_x \, \overline{A}_{x+20} = \left(\frac{\mu^{(\tau)}}{\mu^{(\tau)} + \delta} \right) \left(1 - e^{-20(\mu^{(\tau)} + \delta)} \right)$$

$$\overline{A}^{\,1}_{x:\overline{20}|} = \frac{0.025}{0.075} \left(1 - e^{-20(0.075)} \right) = 0.259$$

(This one could also have been done from scratch using an integral.)
For $\overline{A}^{\,acc.\,1}_{x:\overline{20}|}$, it is easiest to do the integral:

$$\overline{A}^{\,acc.\,1}_{x:\overline{20}|} = \int_0^{20} {}_tp_x^{(\tau)} \; v^t \; \mu_x^{(acc.)}(t)\, dt = \int_0^{20} e^{-0.025t} e^{-0.05t} (0.005)\, dt$$

$$= 0.005 \int_0^{20} e^{-0.075t}\, dt = 0.0518$$

$$\Rightarrow \overline{A}_{\stackrel{1}{x}:\overline{20|}} + \overline{A}^{acc.}_{\stackrel{1}{x}:\overline{20|}} = 0.3108$$

b) The variance is tricky. Let Z be the present value random variable for the benefit. We already found $\mathrm{E}[Z]$ above and

$$\mathrm{Var}[Z] = \mathrm{E}[Z^2] - (\mathrm{E}[Z])^2 .$$

$$Z^2 = \begin{cases} 4v^{2T} & \text{if death is by accident} \\ v^{2T} & \text{if death is by other means} \end{cases}$$

$$\begin{aligned}
\mathrm{E}[Z^2] &= \int_0^{20} {}_tp_x^{(\tau)} \; v^{2t} \; \mu_x^{(2)}(t)\, dt + 4 \int_0^{20} {}_tp_x^{(\tau)} \; v^{2t} \; \mu_x^{(2)}(t)\, dt \\
&= \int_0^{20} e^{-0.025t} \cdot e^{-0.1t}(0.02)\, dt + \int_0^{20} e^{-0.125t}\,(0.005)dt \\
&= \frac{0.02}{0.125}(1 - e^{-2.5}) + \frac{0.02}{0.125}(1 - e^{-2.5}) = 0.2937 \\
\mathrm{Var}[Z] &= 0.2937 - (0.3108)^2 = 0.1971
\end{aligned}$$

Actuarial Mathematics: Chapter 15 – Suggested Problems and Solutions.

Suggested Problems:

5, 6, 10

Suggested Solutions:

5.

$$G\ddot{a}_{[40]:\overline{25}|} = 0.35G + 0.05G\ddot{a}_{[40]:\overline{10}|} + 0.02G\ddot{a}_{[40]:\overline{25}|} + 8.5 + 4\ddot{a}_{[40]:\overline{25}|} + 1000\overline{A}_{[40]:\overline{25}|}$$

$$G = \frac{8.5 + 4\ddot{a}_{[40]:\overline{25}|} + 1000\overline{A}_{[40]:\overline{25}|}}{0.98\ddot{a}_{[40]:\overline{25}|} - 0.35 - 0.05\ddot{a}_{[40]:\overline{10}|}}$$

6.

$$G = 0.04G + 0.025G + 2.5 + 2.5\ddot{a}_{x:\overline{n}|} + 1000\overline{A}_{x:\overline{n}|}$$

$$G = \frac{2.5 + 2.5\ddot{a}_{x:\overline{n}|} + 1000\overline{A}_{x:\overline{n}|}}{0.935}$$

---

10. Assume the face amount is $b \cdot 1000$.
a) First year policy fee satisfies

$$f_1(0.7) = 10 \Rightarrow f_1 = 14.29$$

In Renewal years, the policy fee satifies

$$f_r(0.95) = 2.5 \Rightarrow f_r = 2.63$$

These will be paid separately by first year and renewal. The rest of the annual premium F must satisfy

$$F \cdot \overline{a}_x = 0.25F + 0.05F \cdot \overline{a}_x + 1000b\overline{A}_x + 0.5b\overline{a}_x + 2.5b$$

$$F = \frac{1000b\overline{A}_x + 0.5b\overline{a}_x + 2.5b}{0.95\overline{a}_x - 0.25}$$

Now, the first year premium is $G_1 = F + 14.29$ and the renewal premium is $G_2 = F + 2.63$.

b)

$$f\overline{a}_x = 0.25f + 0.05f\overline{a}_x + 7.5 + 2.5\overline{a}_x$$
$$f = \frac{7.5 + 2.5\overline{a}_x}{0.95\overline{a}_x - 0.25}$$