Probability Models: Chapter 5 – Suggested Problems and Solutions.

Suggested Problems:

(Numbers outside parentheses refer to Edition 7 of Probability Models, numbers in parentheses refer to Edition 8.)

35 (37), 36 (38), 37 (39), 40 (42), 54 (57), 55 (58), 57 (60), 69 (77), 70 (78), 77 (85), 80 (88)

Suggested Solutions:

35 (37).

$$Pr\{N(t+s) - N(s) = n\} = e^{-\lambda t}\frac{(\lambda t)^n}{n!}$$

In this question, $\lambda = 3$ per minute, $s = 0$ and t varies as specified.
$n = 0$ since we're looking for the probability of survival which requires 0 cars coming during the crossing of the road.

$$Pr\{N(t+0) - N(0) = 0\} = e^{-3t}\frac{(3t)^0}{0!} = e^{-3t}$$

$$t = 2 \text{ sec } = 0.0333 \text{ min } \rightarrow 0.9048$$

$$t = 5 \text{ sec } = 0.0833 \text{ min } \rightarrow 0.7788$$

$$t = 10 \text{ sec } = 0.1666 \text{ min } \rightarrow 0.6065$$

$$t = 20 \text{ sec } = 0.333 \text{ min } \rightarrow 0.3679$$

We need the result for $t = 30$ sec in the next problem, so we'll go ahead and get it here:

$$t = 30 \text{ sec } = 0.5 \text{ min } \rightarrow 0.2231$$

Make sure you use the correct units in these type problems (sec vs. min). In addition, make sure your results make sense. For example, it makes sense that if it takes 30 sec to cross the road, the likelihood of being uninjured should be low!

---

36 (38). This questions also seeks the probability of no injury while crossing, but 1 car while crossing will not result in injury. So, we need the probability that either

0 or 1 cars come during the crossing period. We have the probabilities of 0 from the previous question, so we only need to find:

$$Pr\{N(t+0) - N(0) = 1\} = e^{-3t}\frac{(3t)^1}{1!}$$

$$= e^{-3t}\,(3t)$$

So, we have

$$t = 5 \text{ sec } = 0.0833 \text{ min } \rightarrow 0.1947$$

$$t = 10 \text{ sec } = 0.1666 \text{ min } \rightarrow 0.3033$$

$$t = 20 \text{ sec } = 0.333 \text{ min } \rightarrow 0.3679$$

$$t = 30 \text{ sec } = 0.5 \text{ min } \rightarrow 0.3347$$

Totaling the probability of 0 and 1 cars for each time period gives:

$$t = 5 \text{ sec } \rightarrow 0.7788 + 0.1947 = 0.9735$$

$$t = 10 \text{ sec } \rightarrow 0.3033 + 0.6065 = 0.9098$$

$$t = 20 \text{ sec } \rightarrow 0.3679 + 0.3679 = 0.7358$$

$$t = 30 \text{ sec } \rightarrow 0.3347 + 0.2231 = 0.5578$$

---

37 (39). We have $\lambda = 2.5/yr$, so

$$S_{196}\ \tilde{\text{G}}\text{amma}(196, 2.5)$$

$$\text{(a) } E(S_{196}) = \frac{196}{2.5} = 78.4$$

$$\text{(b) } V(S_{196}) = \frac{196}{2.5^2} = 31.36$$

$$Pr\,(T_{196} \leq 67.2) = Pr\,(S_{196} < 67.2) = F_{S_{196}}(67.2) \text{ for (c)}$$

$$\text{(d) } 1 - F_{S_{196}}(90)$$

$$\text{(e) } 1 - F_{S_{196}}(100)$$

Numerical approximations for c), d), and e) could be obtained using the normal distribution with mean $\mu = 78.4$ and $\sigma = 5.6$.

---

40 (42). a) $\text{E}[S_4] = \frac{4}{\lambda}$

b) We can think of this as a new Poisson Process beginning at $t = 1$ for which we are seeking S_2. Answer:

$$1 + \frac{2}{\lambda}$$

c) What happened at time $t = 1$ is irrelevant for the expected number of events between time $t = 2$ and time $t = 4$. The expected number of events for that time period of length 2 is the same as $\mathrm{E}[N(2)] = \frac{2}{\lambda}$.

---

54 (57).

$$\lambda = 2 \text{ per hour}$$

(a)

$$Pr\{N(8+1) - N(8) = 0\} = e^{-2(1)}\frac{2(1)^0}{0!} = 0.1353$$

(b)

$$E(S_4) = \frac{4}{2} = 2$$

And, $12 + 2 = 2$ p.m.

(c) The probability of two or more is one minus the probability of either zero or one.

$$Pr\{N(6+2) - N(6) = 0\} = e^{-2(2)}\frac{2(2)^0}{0!} = e^{-4}$$

$$Pr\{N(6+2) - N(6) = 1\} = e^{-2(2)}\frac{2(2)^1}{0!} = 4e^{-4}$$

So the probability we seek is:

$$1 - \left(e^{-4} + 4e^{-4}\right) = 1 - 5e^{-4}$$

---

55 (58). The set of pulses that arrive at the Geiger counter <u>and</u> are counted forms a Poisson proces with rate $\lambda = 2$.

(a) $Pr\{X(t) = 0\} = e^{-2t}$

(b) $E[X(t)] = 2t$

---

57 (60). a)

$$\left(\frac{20}{60}\right)^2 = \frac{1}{9}$$

b)

$$1 - \Pr[\text{both in last 40 minutes}] = 1 - \frac{4}{9} = \frac{5}{9}$$

69 (77).

$$\Pr\{N(5) - N(4) = n\} = e^{m(5)-m(4)} \frac{[m(5) - m(4)]^n}{n!}$$

$$m(5) = 35 \qquad m(4) = 24$$

$$\Rightarrow \Pr\{N(5) - N(4) = n\} = e^{11} \frac{[11]^n}{n!}$$

---

70 (78).

$$\Pr\{N(9) = n\} = e^{-m(9)} \frac{[m(9)]^n}{n!}$$

$$m(9) = \int_0^9 \lambda(t)\, dt$$

$$\lambda(t) = \begin{cases} 4 & 0 \le t \le 2 \\ 8 & 2 < t \le 4 \\ 4 + t & 4 < t \le 6 \\ 22 - 2t & 6 < t \le 9 \end{cases}$$

$$\Rightarrow m(9) = \int_0^2 4\, dt + \int_2^4 8\, dt + \int_4^6 (4 + t)\, dt + \int_6^9 (22 - 2t)\, dt$$

$$= 8 + 16 + 18 + 21 = 63$$

$$\Rightarrow \Pr\{N(9) = n\} = e^{-63} \frac{[63]^n}{n!}$$

---

77 (85).

$$\mathrm{E}[X(t)] = \mathrm{E}[N(t)] \cdot \mathrm{E}[Y_1] = (5 \cdot 4) \cdot (2000) = 40000.$$

$$\mathrm{Var}[X(t)] = \mathrm{E}[N(t)]\,\mathrm{Var}\,(Y_1) + \mathrm{Var}\,(N(t))\,(\mathrm{E}\,[Y_1])^2$$

$$= (20) \cdot (4{,}000{,}000) + (20) \cdot (2000)^2 = (40)(4{,}000{,}000) = 160{,}000{,}000$$

---

80 (88). Although they don't say so, the idea is to use a Normal distribution with the same mean and variance as the total loss distribution.

$$\mathrm{E}[X(t)] = \mathrm{E}[N(t)] \cdot \mathrm{E}[Y_1] = (15 \cdot 12)(30) = 5400$$

$$\mathrm{Var}[X(t)] = \mathrm{E}[N(t)]\,\mathrm{Var}\,(Y_1) + \mathrm{Var}\,(N(t))\,(\mathrm{E}\,[Y_1])^2$$

$$= (15 \cdot 12)(2500) + (15 \cdot 12)(900) = 450000 + 162000 = 612000$$

So the standard deviation is 782.3.

$$\Pr[X(t) < 6000] = \Pr\left[N(0,1) < \frac{6000 - 5400}{782.3}\right] = \Phi(0.77) \approx 0.779$$